Campbell-like Theorems for Nonhomogeneous Batch Markovian Arrival Processes

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Abstract

We introduce various extensions of Campbell's Theorem that hold for an arbitrary nonhomogeneous Batch Markovian Arrival Process. Both the Laplace functional, and various moments associated with such Batch Markovian Arrival Processes are analyzed in detail. Applications of these ideas to the study of infinite-server queues are also discussed.

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1 Introduction

The main objective of this study is to show how Campbell's Theorem, an important result from the theory of Poisson processes, can be extended to the more general class of nonhomogeneous Batch Markovian Arrival Processes (BMAPs). After we state and prove our extensions, we will then explain how these ideas relate to the study of infinite-server queues fed by nonhomogeneous BMAPs.

Readers should recall that a point process $\{N(t); t \geq 0\}$ on $[0, \infty)$ has associated with it a nondecreasing sequence of random variables $\{T_n\}_{n\geq 1}$ that satisfy the following properties: for each $t \geq 0$,

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function, equal to 1 if $(\cdot)$ is true, and 0 if $(\cdot)$ is false. Furthermore, for each integer $n \geq 1$,

$$T_n = \inf\{t \geq 0 : N(t) \geq n\}.$$

These two relationships tell us there is a one-to-one correspondence between the points $\{T_n\}_{n\geq 1}$ and the sample path $\{N(t); t \geq 0\}$.

The Laplace functional of a point process N is a functional $L_N : \mathcal{C}_K^+ \to \mathbb{R}$, where $\mathcal{C}_K^+$ is the set of all nonnegative, continuous functions defined on $[0, \infty)$, that vanish outside of a compact set and map to the real line. In particular,

$$L_N(g) := \mathbb{E}[e^{-\int_{[0,\infty)} g(s)N(ds)}], \quad g \in \mathcal{C}_K^+$$

where the integral appearing in the definition of L_N is interpreted as a Lebesgue-Stieltjes integral. Moreover,

$$\int_{[0,\infty)} g(s)N(ds) = \sum_{n=1}^{\infty} g(T_n).$$

The Laplace functional is a useful object because two point processes are equal in distribution if and only if their Laplace functionals are equal. It is possible to study the Laplace functional when its domain is larger than $\mathcal{C}_K^+$, but the finite-dimensional distributions of N are completely determined by its Laplace functional when the functional is only defined on $\mathcal{C}_K^+$. Furthermore, in many of our derivations, this assumption will allow us to avoid having to make statements that are only true 'almost-surely' with respect to Lebesgue measure.

Instead of studying the Laplace functional directly, we will instead study the related quantities

$$L_N(g,t) := \mathbb{E}[e^{-\int_{[0,t]} g(s)N(ds)}], \qquad g \in \mathcal{C}_K^+, \qquad t \geq 0.$$

Clearly, for each $g \in \mathcal{C}_K^+$, the dominated convergence theorem yields

$$L_N(g) = \lim_{t\to\infty} L_N(g,t)$$

and in fact, there exists a real number $t_0 \geq 0$ (depending on g) such that $L_N(g,t) = L_N(g)$ for each $t \geq t_0$.

It is well-known that $L_N(g,t)$ admits a simple form when N is a nonhomogeneous Poisson process, having rate function $\lambda : [0,\infty) \to [0,\infty)$: we assume for convenience that λ is continuous on $[0,\infty)$, but this assumption can be relaxed in various ways. Associated with λ is the mean measure Λ defined on $\mathcal{B}([0,\infty))$, the Borel sets of $[0,\infty)$, where for each $A \in \mathcal{B}([0,\infty))$,

$$\Lambda(A) := \int_A \lambda(s)ds.$$

The following statement addresses a version of Campbell's Theorem for nonhomogeneous Poisson processes on $[0,\infty)$.

Theorem 1.1. *(Campbell's Theorem) For each real $t \geq 0$, and each $g \in \mathcal{C}_K^+$, we have*

$$\mathbb{E}[e^{-\int_{[0,t]} g(s)N(ds)}] = e^{-\int_0^t (1-e^{-g(s)})\lambda(s)ds}. \tag{1}$$

Moreover,

$$\mathbb{E}\left[\int_{[0,t]} g(s)N(ds)\right] = \int_0^t g(s)\lambda(s)ds \tag{2}$$

and

$$Var\left(\int_{[0,t]} g(s)N(ds)\right) = \int_0^t g(s)^2\lambda(s)ds. \tag{3}$$

Equation (1) can be established through conditioning on $N(t)$, then interpreting the point locations in $[0,t]$ as order statistics of i.i.d. random variables: see e.g. Chapter 3 of Serfozo [8]. Likewise, as is done in [8], replacing g with αg, where $\alpha \in [0,\infty)$, yields the Laplace transform of the random variable

$$\int_{[0,t]} g(s)N(ds)$$

from which both (2) and (3) can be established through differentiation with respect to α.

Our objective is to show how a combination of the Campbell-Mecke formula and the Silvnyak-Mecke Theorem from the theory of Palm distributions (see e.g. Baccelli et al. [1] or Kallenberg [4]) can be used to establish a version of Campbell's Theorem for nonhomogeneous BMAPs. This type of result, to the best of the author's knowledge, has not yet been observed even for homogeneous MAPs, although the moment recursions we derive are very similar to the moment recursions derived in Nielsen et al. [6], where the authors focus on studying moments of the counting process $N(t)$ associated with the BMAP. Closely related ideas were used recently in previous work of the author [3] which focused on an analysis of a class of stochastic gene expression models: in fact, it could be argued that this paper was inspired by [3]. Once we establish a version of Campbell's theorem for a nonhomogeneous BMAP, we will then examine how these ideas can be used to study the joint Laplace functional of a finite number of thinned nonhomogenous MAPs, all thinned from a single underlying MAP.

A second objective of this paper is to show what happens when related ideas are used to study an infinite-server queue fed by a nonhomogeneous BMAP. The point process arguments we use to study one set of random variables associated with this queue can be argued to be somewhat analogous to the type of argument used to derive the Kolmogorov backward equations of a countable-state continuous-time Markov chain (CTMC), while the arguments we use to study a different, yet related set of random variables are analogous to the type of argument used to derive the Kolmogorov forward equations of a CTMC. The arguments we use to establish Campbell's Theorem for a BMAP are analogous to the type of argument used to establish the Kolmogorov forward equations of a countable-state CTMC, and given the framework of Campbell's theorem, it turns out that the forward approach is the most useful approach. This should be clearer to readers once our Campbell-like results are compared to the results we derive for infinite-server queues fed by a nonhomogeneous BMAP.

We close the introduction by mentioning that all random elements discussed in this study are assumed to exist on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathcal{A}$ is the Borel σ-field generated by the complete, separable metric space (Ω, d) having metric d. While this may at first sound like a major restriction, all random elements we will encounter can be constructed on such a space, and we need the underlying probability space to have this structure in order to guarantee various Palm measures we use both exist, and are unique in some sense. These issues will not play a major role in our study in any way, but we refer readers interested in seeing these details to Chapters 10 and 11 of Kallenberg [4].

2 Extending Campbell's Theorem to Nonhomogeneous Batch Markovian Arrival Processes

A nonhomogeneous BMAP is governed by a continuous-time Markov chain $\{X(t); t \geq 0\}$ having state space $S = \mathbb{N} \times E$, where $\mathbb{N}$ denotes the set of all nonnegative integers, and $E := \{1, 2, \ldots, m\}$ for some integer $m \geq 1$. It helps to write S as

$$S := \bigcup_{n=0}^{\infty} L_n$$

where for each $n \in \mathbb{N}$,

$$L_n := \{(n, 1), (n, 2), \ldots, (n, m)\}$$

represents level n. We can express $X(t)$ as $X(t) = (N(t), I(t))$, where $N(t)$ denotes the level visited by the chain at time t, and $I(t)$ the phase of the chain at time t. The process $\{N(t); t \geq 0\}$ is the main object of interest in our study, and we refer to it as the counting process of the BMAP.

The CTMC $\{X(t); t \geq 0\}$ has associated with it a family of transition matrices $\{\boldsymbol{\mathcal{Q}}(t); t \geq 0\}$, where for each $t \geq 0$, the rows and columns of $\boldsymbol{\mathcal{Q}}(t)$ are arranged based on a natural lexicographical ordering of the states in S: given two states $(n_1, i_1), (n_2, i_2) \in S$, we say $(n_1, i_1) < (n_2, i_2)$ if either (a) $n_1 < n_2$, or (b) $n_1 = n_2$ and $i_1 < i_2$. Using this ordering scheme, each transition matrix $\boldsymbol{\mathcal{Q}}(t)$ is of the form

$$\boldsymbol{\mathcal{Q}}(t) = \begin{pmatrix} \mathbf{K}_0(t) & \mathbf{K}_1(t) & \mathbf{K}_2(t) & \mathbf{K}_3(t) & \mathbf{K}_4(t) & \ldots \\ \mathbf{0} & \mathbf{K}_0(t) & \mathbf{K}_1(t) & \mathbf{K}_2(t) & \mathbf{K}_3(t) & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_0(t) & \mathbf{K}_1(t) & \mathbf{K}_2(t) & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_0(t) & \mathbf{K}_1(t) & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_0(t) & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

For each $t \geq 0$, the matrices $\mathbf{K}_\ell(t)$, $\ell \geq 0$ are elements of $\mathbb{R}^{m \times m}$, and $\mathbf{0} \in \mathbb{R}^{m \times m}$ denotes the zero matrix. Each matrix $\mathbf{K}_\ell(t)$, for $\ell \geq 1$ is a $m \times m$ matrix whose elements are all nonnegative, and $\mathbf{K}_0(t)$ is a matrix whose off-diagonal elements are all nonnegative, and its diagonal elements are nonpositive, and defined in a manner so that each row sum of $\boldsymbol{\mathcal{Q}}(t)$ is equal to zero. More particularly, for each integer $\ell \geq 0$,

$$\mathbf{K}_\ell(t) = [k_{\ell;z,w}(t)]_{z,w \in S}$$

where for each $z \in E$,

$$k_{0;z,z}(t) = -\sum_{w \neq z} k_{0;z,w}(t) - \sum_{\ell \geq 1} \sum_{w \in E} k_{\ell;z,w}(t).$$

We assume each function $k_{\ell;z,w} : [0, \infty) \to [0, \infty)$ is continuous on $[0, \infty)$.

The proof techniques we use throughout will involve making use of how $\{X(t); t \geq 0\}$ can be constructed with a countable collection of nonhomogeneous Poisson processes. For each $z, w \in E$ satisfying $z \neq w$, let $A^{(0)}_{z,w}$ represent a nonhomogeneous Poisson process on $[0, \infty)$ having rate function $k_{0;z,w}$, and for each integer $\ell \geq 1$, and each $z, w \in S$ (where possibly $z = w$), let $A^{(\ell)}_{z,w}$ represent a nonhomogeneous Poisson process on $[0, \infty)$ having rate parameter $k_{\ell;z,w}$. We assume throughout that the aforementioned nonhomogeneous Poisson processes are independent. Finally, for each phase $y \in S$, we define the Poisson process $A^{(0)}_y$ as

$$A^{(0)}_y := \sum_{z \neq y} A^{(0)}_{y,z} + \sum_{\ell=1}^{\infty} \sum_{z} A^{(\ell)}_{y,z}.$$

Together these Poisson processes can be used to construct a sample path of $\{X(t); t \geq 0\}$ in a way that is best illustrated through the following equality: for each state $y \in E$, and each integer $n \geq 0$,

$$\begin{aligned} \mathbf{1}_{\{X(t)=(n+1,y)\}} &= \sum_{\ell=1}^{n+1} \sum_{z \in S} \int_{[0,t]} \mathbf{1}_{\{X(s-)=(n+1-\ell,z)\}} \mathbf{1}_{\{A^{(0)}_y(s,t]=0\}} A^{(\ell)}_{z,y}(ds) \\ &\quad + \sum_{z \in S: z \neq y} \int_{[0,t]} \mathbf{1}_{\{X(s-)=(n+1,z)\}} \mathbf{1}_{\{A^{(0)}_y(s,t]=0\}} A^{(0)}_{z,y}(ds) \end{aligned}$$

and if $X(0) = (0, y_0)$,

$$\begin{aligned} \mathbf{1}_{\{X(t)=(0,y)\}} &= \mathbf{1}_{\{y=y_0\}} \mathbf{1}_{\{A^{(0)}_y(0,t]=0\}} \\ &\quad + \sum_{z \in S, z \neq y} \int_{[0,t]} \mathbf{1}_{\{X(s-)=(0,z)\}} \mathbf{1}_{\{A^{(0)}_y(s,t]=0\}} A^{(0)}_{z,y}(ds). \end{aligned}$$

Readers wishing to see a more rigorous construction of $\{X(t); t \geq 0\}$ from these Poisson processes (for the homogeneous case), including a proof showing $\{X(t); t \geq 0\}$ is a CTMC under this construction, are referred to Chapter 9 of Brémaud [2].

2.1 Campbell's Theorem for a Nonhomogeneous BMAP

For each $t \in [0, \infty)$ and each $g \in \mathcal{C}^+_K$, we define the matrix $\mathbf{L}(t, g)$ as

$$\mathbf{L}(t, g) := \left[\mathbb{E}_x \left[e^{-\int_{[0,t]} g(s) N(ds)} \mathbf{1}_{\{I(t)=y\}} \right] \right]_{x,y \in E}.$$

Throughout, we let $\mathbb{P}_x$ denote a conditional probability, where we condition on $I(0) = x$, and we let $\mathbb{E}_x$ denote the conditional expectation associated with $\mathbb{P}_x$.

Our first result is a natural BMAP extension of Campbell's Theorem.

Theorem 2.1. *The matrices $\mathbf{L}(t, g)$, for $t \geq 0$ and $g \in \mathcal{C}^+_K$, satisfy the following ordinary differential equation: for each $t > 0$,*

$$\frac{\partial}{\partial t} \mathbf{L}(t, g) = \mathbf{L}(t, g) \sum_{\ell=0}^{\infty} \mathbf{K}_\ell(t) e^{-\ell g(t)}$$

where $\mathbf{L}(0, g) = \mathbf{I}$.

Proof. Fix an arbitrarily chosen state $x \in E$, and set $X(0) = (0, x)$. Observe that for each $y \in S$, where possibly $y = x$,

$$\begin{aligned}
& e^{-\int_{[0,t]} g(s)N(ds)} \mathbf{1}_{\{I(t)=y\}} \\
&= \mathbf{1}_{\{x=y\}} \mathbf{1}_{\{A_y^{(0)}(0,t]=0\}} \\
&\quad + \sum_{z \neq y} \int_{(0,t]} e^{-\int_{[0,s)} g(u)N(du)} \mathbf{1}_{\{I(s-)=z\}} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} A_{z,y}^{(0)}(ds) \\
&\quad + \sum_{\ell \geq 1} \sum_{z} \int_{(0,t]} e^{-\int_{[0,s)} g(u)N(du)} \mathbf{1}_{\{I(s-)=z\}} e^{-\ell g(s)} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} A_{z,y}^{(\ell)}(ds).
\end{aligned} \tag{4}$$

Taking the expected value of both sides, while applying the Campbell-Mecke formula repeatedly to the right-hand-side of (4) yields

$$\begin{aligned}
& \mathbb{E}_x[e^{-\int_{[0,t]} g(s)N(ds)} \mathbf{1}_{\{I(t)=y\}}] \\
&= \mathbf{1}_{\{x=y\}} e^{\int_0^t k_{0;y,y}(s)ds} \\
&\quad + \sum_{z \neq y} \int_0^t \mathcal{E}_{x,s}^{(0;z,y)} \left[e^{-\int_{[0,s)} g(u)N(du)} \mathbf{1}_{\{I(s-)=z\}} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} \right] k_{0;z,y}(s)ds \\
&\quad + \sum_{\ell \geq 1} \sum_{z} \int_0^t \mathcal{E}_{x,s}^{(\ell;z,y)} \left[e^{-\int_{[0,s)} g(u)N(du)} \mathbf{1}_{\{I(s-)=z\}} e^{-\ell g(s)} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} \right] k_{\ell;z,y}(s)ds
\end{aligned} \tag{5}$$

where $\{\mathcal{P}_{x,s}^{(\ell;z,y)}\}_{s \geq 0}$ is the collection of Palm measures induced by the Poisson process $A_{z,y}^{(\ell)}$. Next, recall that by the Silvnyak-Mecke theorem, the law of $\mathcal{P}_{x,s}^{(\ell;z,y)}$ on $\mathcal{A}$ is the same as the law of $\mathbb{P}_x$ on $\mathcal{A}$, except $A_{z,y}^{(\ell)}$ has a point at location s with probability one. Thus,

$$\begin{aligned}
\mathcal{E}_{x,s}^{(\ell;z,y)} \left[e^{-\int_{[0,s)} g(u)N(du)} \mathbf{1}_{\{I(s-)=z\}} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} \right] &= \mathbb{E}_x \left[e^{-\int_{[0,s)} g(u)N(du)} \mathbf{1}_{\{I(s-)=z\}} \right] \mathbb{E}_y \left[\mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} \right] \\
&= \mathbb{E}_x \left[e^{-\int_{[0,s]} g(u)N(du)} \mathbf{1}_{\{I(s)=z\}} \right] e^{\int_s^t k_{0;y,y}(u)du}.
\end{aligned}$$

Plugging these observations into (5) then gives

$$\begin{aligned}
& \mathbb{E}_x[e^{-\int_{[0,t]} g(s)N(ds)} \mathbf{1}_{\{I(t)=y\}}] \\
&= \mathbf{1}_{\{x=y\}} e^{\int_0^t k_{0;y,y}(u)du} \\
&\quad + \sum_{z \neq y} \int_0^t \mathbb{E}_x[e^{-\int_{[0,s]} g(u)N(du)} \mathbf{1}_{\{I(s)=z\}}] k_{0;z,y}(s) e^{\int_s^t k_{0;y,y}(u)du} ds \\
&\quad + \sum_{\ell \geq 1} \sum_{z} \int_0^t \mathbb{E}_x[e^{-\int_{[0,s]} g(u)N(du)} \mathbf{1}_{\{I(s)=z\}}] e^{-\ell g(s)} k_{\ell;z,y}(s) e^{\int_s^t k_{0;y,y}(u)du} ds.
\end{aligned} \tag{6}$$

After multiplying both sides of (6) by $\exp\{-\int_0^t k_{0;y,y}(u)du\}$, taking derivatives with respect to t, and re-expressing the resulting equations in matrix form, we get

$$\frac{\partial}{\partial t} \mathbf{L}(t,g) = \mathbf{L}(t,g)\mathbf{K}_0(t) + \mathbf{L}(t,g) \sum_{\ell \geq 1} e^{-\ell g(t)} \mathbf{K}_\ell(t) = \mathbf{L}(t,g) \sum_{\ell=0}^{\infty} e^{-\ell g(t)} \mathbf{K}_\ell(t)$$

proving the claim. □

We now study various moments associated with a MAP. Given a function $g \in \mathcal{C}_K^+$ we define, for each integer $n \geq 0$ and each real number $t \geq 0$, the matrix

$$\mathbf{M}_n(g,t) := \left[\mathbb{E}_x \left[\left(\sum_{\ell=1}^{N(t)} g(T_\ell) \right)^n \mathbf{1}_{\{I(t)=y\}} \right] \right]_{x,y \in E}.$$

The next result shows that the matrices $\mathbf{M}_n(g,t)$ satisfy a recursive system of linear ordinary differential equations.

Theorem 2.2. *Given an integer $m \geq 0$, suppose that for each integer $k \in \{0, 1, 2, \dots, m+1\}$, the matrix function*

$$\sum_{\ell \geq 1} (\ell g(t))^k \mathbf{K}_\ell(t)$$

is finite for each $t \in [0, \infty)$, and is continuous on $[0, \infty)$. Then for each $n \in \{0, 1, 2, \dots, m\}$,

$$\frac{\partial}{\partial t} \mathbf{M}_{n+1}(g,t) = \mathbf{M}_{n+1}(g,t) \sum_{\ell \geq 0} \mathbf{K}_\ell(t) + \sum_{k=0}^{n} \binom{n+1}{k} \mathbf{M}_k(g,t) \sum_{\ell \geq 1} (\ell g(t))^{n+1-k} \mathbf{K}_\ell(t). \quad (7)$$

Proof. Fix an arbitrary state $x \in E$. Then for each integer $n \geq 0$, and each state $y \in E$ (where possibly $y = x$)

$$\begin{aligned}
&\left(\int_{[0,t]} g(u) N(du) \right)^{n+1} \mathbf{1}_{\{I(t)=y\}} \\
&= \sum_{z \neq y} \int_{[0,t]} \left(\int_{[0,s)} g(u) N(du) \right)^{n+1} \mathbf{1}_{\{I(s-)=z\}} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} A_{z,y}^{(0)}(ds) \\
&\quad + \sum_{\ell \geq 1} \sum_{z} \int_{[0,t]} \left(\int_{[0,s)} g(u) N(du) + \ell g(s) \right)^{n+1} \mathbf{1}_{\{I(s-)=z\}} \mathbf{1}_{\{A_y^{(0)}(s,t]=0\}} A_{z,y}^{(\ell)}(ds).
\end{aligned}$$

Taking the expected value of both sides, while applying the Campbell-Mecke formula and the Silvnyak-Mecke theorem to the right hand side gives

$$\begin{aligned}
&\mathbb{E}_x \left[\left(\int_{[0,t]} g(u) N(du) \right)^{n+1} \mathbf{1}_{\{I(t)=y\}} \right] \\
&= \sum_{z \neq y} \int_{[0,t]} \mathbb{E}_x \left[\left(\int_{[0,s]} g(u) N(du) \right)^{n+1} \mathbf{1}_{\{I(s)=z\}} \right] k_{0;z,y}(s) e^{\int_s^t k_{0;y,y}(u) du} ds \\
&\quad + \sum_{\ell \geq 1} \sum_{z} \int_{[0,t]} \mathbb{E}_x \left[\left(\int_{[0,s]} g(u) N(du) + \ell g(s) \right)^{n+1} \mathbf{1}_{\{I(s)=z\}} \right] k_{\ell;z,y}(s) e^{\int_s^t k_{0;y,y}(u) du} ds.
\end{aligned}$$

After multiplying both sides by $e^{-\int_0^t k_{0;y,y}(u) du}$, taking derivatives, and simplifying, we get

$$\frac{\partial}{\partial t} \mathbf{M}_{n+1}(g,t) = \mathbf{M}_{n+1}(g,t) \sum_{\ell \geq 0} \mathbf{K}_\ell(t) + \sum_{k=0}^{n} \binom{n+1}{k} \mathbf{M}_k(g,t) \sum_{\ell \geq 1} \left[(\ell g(t))^{n+1-k} \mathbf{K}_\ell(t) \right]$$

proving the claim. $\square$

It does not appear to be possible to simplify (7) further in general, but it can be simplified when $\sum_{\ell \geq 0} \mathbf{K}_\ell(t)$ is the same for all $t \geq 0$. This condition is satisfied, for instance, when all of the off-diagonal elements of $\mathbf{K}_0(t)$ are constant with respect to t, and when, for each integer $\ell \geq 1$, $\mathbf{K}_\ell(t)$ is a diagonal matrix for each $t \geq 0$: such a nonhomogeneous BMAP can be thought of as a batch Markov-modulated nonhomogeneous Poisson process, where the arrival rate function varies in accordance to a time-homogeneous CTMC having transition rate matrix $\sum_{\ell \geq 0} \mathbf{K}_\ell(0)$.

Corollary 2.1. *Suppose that for each $t \geq 0$, the matrix $\mathbf{K}_0(t) + \mathbf{K}_1(t)$ is invariant with respect to t, and that the hypothesis of Theorem 2.2 is satisfied for some integer $m \geq 0$. Then*

$$\mathbf{M}_1(g,t) = \int_0^t \mathbf{e}^{(\sum_{\ell \geq 0} \mathbf{K}_\ell(0))s} \left[\sum_{\ell \geq 1} (\ell g(s)) \mathbf{K}_\ell(s) \right] e^{(\sum_{\ell \geq 0} \mathbf{K}_\ell(0))(t-s)} ds.$$

and for each integer $1 \leq n \leq m$,

$$\mathbf{M}_{n+1}(g,t) = \sum_{k=0}^{n} \binom{n+1}{k} \int_0^t \mathbf{M}_k(g,s) \sum_{\ell \geq 1} \left[(\ell g(s))^{n+1-k} \mathbf{K}_\ell(s) \right] e^{(\sum_{\ell \geq 0} \mathbf{K}_\ell(0))(t-s)} ds.$$

Proof. Starting from the equation

$$\mathbf{M}_{n+1}'(g,t) - \mathbf{M}_{n+1}(g,t) \sum_{\ell \geq 0} \mathbf{K}_\ell(0) = \sum_{k=0}^{n} \binom{n+1}{k} \mathbf{M}_k(g,t) \sum_{\ell \geq 1} (\ell g(t))^{n+1-k} \mathbf{K}_\ell(t)$$

derived in Theorem 2.2, we observe that this linear matrix ODE is easy to solve: indeed, for each $t \geq 0$, we get that for each integer $n \geq 0$

$$\mathbf{M}_{n+1}(t) = \int_0^t \sum_{k=0}^{n} \binom{n+1}{k} \mathbf{M}_k(g,s) g(s)^{n+1-k} \mathbf{K}_1(s) \mathbf{e}^{(\sum_{\ell \geq 0} \mathbf{K}_\ell(0))(t-s)} ds$$

which proves the claim, since in this case $\mathbf{M}_0(g,s) = e^{(\sum_{\ell \geq 0} \mathbf{K}_\ell(0))s}$. □

As a final 'sanity check', we close this subsection by showing how Corollary 2.1 simplifies when $\{N(t); t \geq 0\}$ is a nonhomogeneous Poisson process having rate function $\lambda : [0,\infty) \to [0,\infty)$.

Corollary 2.2. *For each real $t \geq 0$, and each integer $n \geq 0$,*

$$\mathbb{E}[N(t)^{n+1}] = \sum_{k=0}^{n} \binom{n+1}{k} \int_0^t \mathbb{E}[N(s)^k] g(s)^{n+1-k} \lambda(s) ds.$$

From this result, we get

$$\mathbb{E}[N(t)] = \int_0^t g(s) \lambda(s) ds$$

which is clearly (2), and

$$\begin{aligned}
\mathbb{E}[N(t)^2] &= \int_0^t g(s)^2\lambda(s)ds + 2\int_0^t \mathbb{E}[N(s)]g(s)\lambda(s)ds \\
&= \int_0^t g(s)^2\lambda(s)ds + 2\int_0^t\int_0^s g(u)\lambda(u)g(s)\lambda(s)ds \\
&= \int_0^t g(s)^2\lambda(s)ds + \left(\int_0^t g(s)\lambda(s)ds\right)^2 \\
&= \int_0^t g(s)^2\lambda(s)ds + \mathbb{E}[N(t)]^2
\end{aligned}$$

which implies

$$Var(N(t)) = \int_0^t g(s)^2\lambda(s)ds$$

proving (3).

2.2 A Thinning Result

The proof technique we use to establish Theorem 2.1 can be used to say something about thinning properties of Markovian arrival processes. In order to refrain from further obscuring the basic ideas, we consider only the thinning of a nonhomogeneous MAP instead of a nonhomogeneous BMAP, meaning we assume $\mathbf{K}_\ell(t) = \mathbf{0}$ for each integer $\ell \geq 2$.

Suppose we associate with each point of a Markovian arrival process a label, independently of everything else. Letting L_n denote the label associated with the nth point T_n of a MAP, we assume that for each $k \in \{1, 2, \ldots, n\}$,

$$\mathbb{P}(L_n = k \mid T_n = s) = p_k(s), \qquad s \geq 0$$

where $\sum_{k=1}^n p_k(s) = 1$ for each $s \geq 0$. Given this time-dependent labeling scheme, we define the point processes $N_1, N_2, \ldots, N_n$ as

$$N_k(A) := \int_A \mathbf{1}_{\{L(s)=k\}} N(ds), \qquad A \in \mathcal{B}([0,\infty))$$

where $L(s)$ denotes the label associated with the point located at time s. We assume each function $p_k : [0,\infty) \to [0,1]$ is continuous on its domain. We omit the proof, as it can be derived using essentially the same argument as that used to establish Theorem 2.1.

Theorem 2.3. *For each collection of functions $g_1, g_2, \ldots, g_n \in \mathcal{C}_K^+$, we have*

$$\frac{\partial}{\partial t}\mathbf{L}(t, \boldsymbol{g}) = \mathbf{L}(t, \boldsymbol{g})\left[\mathbf{K}_0(t) + \mathbf{K}_1(t)\sum_{k=1}^n e^{-g_k(t)}p_k(t)\right], \qquad t > 0$$

where $\mathbf{L}(0, \boldsymbol{g}) = \mathbf{I}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$.

We do not expect $N_1, N_2, \ldots, N_n$ to be independent in general, but this result can be used to provide another proof of why these processes are independent when N is a nonhomogeneous Poisson process having rate function λ.

3 An Infinite-Server Queue Fed By a Nonhomogeneous BMAP

In this section, we study to what extent the ideas we used to state and prove a version of Campbell's Theorem for nonhomogeneous BMAPs can also be used to study an infinite-server queue fed by a nonhomogeneous BMAP. To the best of the author's knowledge, this model has not yet been studied in the literature, and we will show in this section that this model becomes much more difficult in various ways when the arrival process is a nonhomogeneous BMAP instead of a homogeneous BMAP. The PH/G/∞ queue, a special case of the homogeneous MAP/G/∞ queue where customers arrive in accordance to a renewal process having phase-type interrenewals, was first analyzed in Ramaswami and Neuts [7]. The homogeneous BMAP/G/∞ queue was first analyzed in Masuyama and Takine [5].

Suppose that customers arrive to an infinite-server queueing system in accordance to the inhomogeneous Batch Markovian Arrival Process (BMAP) described in Section 2. Assume each customer brings a generally distributed amount of work with CDF F, independently of everything else, and let $\overline{F}(t) := 1 - F(t)$ for each $t \in \mathbb{R}$. We also assume $F(0) = 0$, meaning each customer brings a positive amount of work with probability one. Finally, we let $B_j(u)$ denote the amount of work brought by the jth customer arriving at time u, and we let $\{B_j\}_{j\geq 1}$ denote a generic i.i.d. sequence of random variables, each having CDF F.

Our primary objective is to study the distribution of $Q(t)$ and $I(t)$, where $Q(t)$ denotes the number of customers present in the system at time t. Unfortunately, contrary to the case where $\{N(t); t \geq 0\}$ is a homogeneous BMAP, this joint distribution cannot be described with a single ODE, but we can still say something about this distribution using the following trick. Fix a real number $t > 0$, and for each $A \in \mathcal{B}([0,t])$ (the Borel subsets of $[0,t]$), define $Q(A,t)$ as the number of customers arriving in A that are still present in the system at time t. Next, we define, for each $s \in [0,t]$, the matrix

$$\mathbf{L}_{f,t}(\alpha, s) := \left[\mathbb{E}_x[e^{-\alpha Q((0,s],t)}\mathbf{1}_{\{I(s)=y\}}]\right].$$

Theorem 3.1. *Fix a real number $t > 0$. For each $\alpha \in \mathbb{C}_+$, we see that for almost every $s \in (0,t)$ with respect to Lebesgue measure,*

$$\frac{\partial}{\partial s}\mathbf{L}_{f,t}(\alpha, s) = \mathbf{L}_{f,t}(\alpha, s)\sum_{\ell=0}^{\infty}(1-(1-e^{-\alpha})\overline{F}(t-s))^{\ell}\mathbf{K}_\ell(s).$$

Furthermore, $\mathbf{L}_{f,t}(\alpha, 0) = \mathbf{I}$.

From an applications standpoint, this system of ODEs is less desirable, because usually the goal is to study $\mathbf{L}_{f,t}(\alpha, t)$ for each $t > 0$.

Proof. First observe that

$$\begin{aligned}
&e^{-\alpha Q((0,s],t)}\mathbf{1}_{\{I(s)=y\}}\\
&= \mathbf{1}_{\{x=y\}}\mathbf{1}_{\{A_y^{(0)}(s)=0\}}\\
&\quad+\sum_{z\neq y}\int_{(0,s]} e^{-\alpha Q((0,u),t)}\mathbf{1}_{\{I(u-)=z\}}\mathbf{1}_{\{A_y^{(0)}(u,s]=0\}}A_{z,y}^{(0)}(du)\\
&\quad+\sum_{\ell=1}^{\infty}\sum_{z}\int_{(0,s]} e^{-\alpha Q((0,u),t)}\mathbf{1}_{\{I(u-)=z\}}e^{-\alpha\sum_{j=1}^{\ell}\mathbf{1}_{\{B_j(u)>t-u\}}}\mathbf{1}_{\{A_y^{(0)}(u,s]=0\}}A_{z,y}^{(\ell)}(du).
\end{aligned}$$

Taking the expected value of both sides, while applying the Campbell-Mecke formula to the right hand side gives

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\alpha Q((0,s],t)}\mathbf{1}_{\{I(s)=y\}}\right]\\
&=\mathbf{1}_{\{x=y\}}e^{\int_0^s k_{0;y,y}(u)du}\\
&\quad+\sum_{z\neq y}\int_0^s \mathbb{E}_x\left[e^{-\alpha Q((0,u],t)}\mathbf{1}_{\{I(u)=z\}}\right]k_{0;z,y}(u)e^{\int_u^s k_{0;y,y}(v)dv}du\\
&\quad+\sum_{k=1}^{\infty}\sum_{z}\int_0^s \mathbb{E}_x\left[e^{-\alpha Q((0,u],t)}\mathbf{1}_{\{I(u)=z\}}\right]\mathbb{E}[e^{-\alpha\sum_{j=1}^k \mathbf{1}(B_j>t-u)}]k_{\ell;z,y}e^{\int_u^s k_{0;y,y}(v)dv}du.
\end{aligned}
$$

Multiplying both sides of the equality by $e^{-\int_0^s k_{0;y,y}(v)dv}$, we get after taking derivatives of both sides with respect to s and simplifying that

$$
\frac{\partial}{\partial s}\mathbf{L}_{f,t}(\alpha,s)=\mathbf{L}_{f,t}(\alpha,s)\mathbf{K}_0(s)+\sum_{\ell=1}^{\infty}\mathbf{L}_{f,t}(\alpha,s)\mathbb{E}[e^{-\alpha\sum_{j=1}^{\ell}\mathbf{1}(B_j>t-s)}]\mathbf{K}_\ell(s)
$$

which proves the claim. □

We next derive a recursion for the moments of $Q((0,s],t)$, for each $s\in[0,t]$. Fix $t>0$, and for each $s\in[0,t]$, define

$$
\mathbf{M}_{f,n}(s):=\left[\mathbb{E}_x\left[Q((0,s],t)^n\mathbf{1}_{\{I(s)=y\}}\right]\right]_{x,y\in E}.
$$

Theorem 3.2. *Fix a real number $t>0$, and assume there exists an integer $m\geq 0$ such that for each $k\in\{0,1,2,\dots,m+1\}$, the function*

$$
\sum_{\ell\geq 1}\mathbb{E}\left[\left(\sum_{j=1}^{\ell}\mathbf{1}_{\{B_j>t-s\}}\right)^k\right]\mathbf{K}_\ell(s)
$$

is finite for each $s\in[0,t]$, and continuous almost-surely (with respect to Lebesgue measure) on $[0,t]$. Then for almost every $s\in(0,t)$ (with respect to Lebesgue measure), we have for $0\leq n\leq m$ that

$$
\frac{\partial}{\partial s}\mathbf{M}_{f,n+1}(s)=\mathbf{M}_{f,n+1}(s)\sum_{\ell\geq 0}\mathbf{K}_\ell(s)+\sum_{k=0}^{n}\binom{n+1}{k}\mathbf{M}_{f,k}(s)\sum_{\ell\geq 1}\mathbb{E}\left[\left(\sum_{j=1}^{\ell}\mathbf{1}_{\{B_j>t-s\}}\right)^{n+1-k}\right]\mathbf{K}_\ell(s). \tag{8}
$$

Proof. The proof is similar to the proof used to establish Theorem 2.2, except we start with the following observation: for each integer $n\geq 0$,

$$
\begin{aligned}
&Q((0,s],t)^{n+1}\mathbf{1}_{\{I(s)=y\}}\\
&=\sum_{z\neq y}\int_{(0,s]}Q((0,u),t)^{n+1}\mathbf{1}_{\{I(u-)=z\}}\mathbf{1}_{\{A_y^{(0)}(u,s]=0\}}A_{z,y}^{(0)}(du)\\
&+\sum_{\ell\geq 1}\sum_{z}\int_{(0,s]}\left(Q((0,u),t)+\sum_{j=1}^{\ell}\mathbf{1}_{\{B_j(u)>t-u\}}\right)^{n+1}\mathbf{1}_{\{I(u-)=z\}}\mathbf{1}_{\{A_y^{(0)}(u,s]=0\}}A_{z,y}^{(\ell)}(du).
\end{aligned}
$$

After taking the expectation of both sides, while applying both the Campbell-Mecke formula and the Silvnyak-Mecke formula, and simplifying accordingly, we arrive at (8). $\square$

An analogous linear ODE can be derived for a different set of quantities. Fix a real number $t > 0$, and for each $s \in [0, t]$, and each $\alpha \in \mathbb{C}_+$, define the matrix

$$\mathbf{L}_{b,t}(\alpha, s) := \left[\mathbb{E}\left[e^{-\alpha Q((s,t],t)}\mathbf{1}_{\{I(t)=y\}} \mid I(s) = x\right]\right]_{x,y\in E}.$$

Theorem 3.3. *Fix a real number $t > 0$. For each $\alpha \in \mathbb{C}_+$, and each $s \in (0, t)$,*

$$\frac{\partial}{\partial s}\mathbf{L}_{b,t}(\alpha, s) = -\left[\sum_{\ell=0}^{\infty}\mathbf{K}_\ell(s)(1 - (1 - e^{-\alpha})\overline{F}(t - s))^\ell\right]\mathbf{L}_{b,t}(\alpha, s).$$

Furthermore, $\mathbf{L}_{b,t}(\alpha, t) = \mathbf{I}$.

Obviously, we have $\mathbf{L}_{b,t}(\alpha, 0) = \mathbf{L}_{f,t}(\alpha, t)$.

Proof. The proof of this result is similar to the proof of Theorem 3.1. For $0 < s < t$,

$$\begin{aligned}
&\mathbf{1}_{\{I(s)=x\}}e^{-\alpha Q((s,t],t)}\mathbf{1}_{\{I(t)=y\}} \\
&= \mathbf{1}_{\{x=y\}}\mathbf{1}_{\{I(s)=x\}}\mathbf{1}_{\{A_y(s,t]=0\}} \\
&+ \mathbf{1}_{\{I(s)=x\}}\sum_{z\neq y}\int_{(s,t]}\mathbf{1}_{\{A_x^{(0)}(s,u)=0\}}e^{-\alpha Q((u,t],t)}\mathbf{1}_{\{I(t)=y\}}A_{x,z}^{(0)}(du) \\
&+ \sum_{\ell\geq 1}\sum_{z}\mathbf{1}_{\{I(s)=x\}}\sum_{z\neq y}\int_{(s,t]}\mathbf{1}_{\{A_x^{(0)}(s,u)=0\}}e^{-\sum_{j=1}^{\nu}\mathbf{1}_{\{B_j(u)>t-u\}}}e^{-\alpha Q((u,t],t)}\mathbf{1}_{\{I(t)=y\}}A_{x,z}^{(\ell)}(du).
\end{aligned}$$

Taking the conditional expectation of both sides, with respect to $\mathbb{E}[\cdot \mid I(s) = x]$, then applying both the Campbell-Mecke formula and the Silvnyak-Mecke formula yields

$$\begin{aligned}
&\mathbb{E}[e^{-\alpha Q((s,t],t)}\mathbf{1}_{\{I(t)=y\}} \mid I(s) = x] \\
&= \mathbf{1}_{\{x=y\}}e^{\int_s^t k_{0;x,x}(u)du} \\
&+ \sum_{z\neq y}\int_s^t e^{\int_s^u k_{0;x,x}(v)dv}k_{0;x,z}(u)\mathbb{E}\left[e^{-\alpha Q((u,t],t)}\mathbf{1}_{\{I(t)=y\}} \mid I(u) = z\right]du \\
&+ \sum_{\ell\geq 1}\sum_{z}\int_s^t e^{\int_s^u k_{0;x,x}(v)dv}k_{\ell;x,z}(u)\mathbb{E}[e^{-\alpha\sum_{j=1}^{\ell}\mathbf{1}_{\{B_j>t-u\}}}]\mathbb{E}\left[e^{-\alpha Q((u,t],t)}\mathbf{1}_{\{I(t)=y\}} \mid I(u) = z\right]du
\end{aligned}$$

After simplifying these equations further and rewriting them in matrix form, we arrive at the claim. $\square$

We conclude the paper by stating a final moment recursion. Fix a real number $t > 0$, and for each $s \in [0, t]$, and each integer $n \geq 1$, define

$$\mathbf{M}_{b,n}(s) := \left[\mathbb{E}\left[Q((s,t],t)^n\mathbf{1}_{\{I(t)=y\}} \mid I(s) = x\right]\right]_{x,y\in E}.$$

Theorem 3.4. *Fix a real number $t > 0$, and assume there exists an integer $m \geq 0$ such that for each $k \in \{0, 1, 2, \dots, m+1\}$, the function*

$$\sum_{\ell \geq 1} \mathbb{E}\left[\left(\sum_{j=1}^{\ell} \mathbf{1}_{\{B_j > t-s\}}\right)^k\right] \mathbf{K}_\ell(s)$$

is finite for each $s \in [0, t]$, and continuous almost-surely (with respect to Lebesgue measure) Then for almost every $s \in (0, t)$ (with respect to Lebesgue measure), we have for $0 \leq n \leq m$ that

$$\frac{\partial}{\partial s}\mathbf{M}_{b,n+1}(s) = -\left[\sum_{\ell \geq 0} \mathbf{K}_\ell(s)\right] \mathbf{M}_{b,n+1}(s) - \sum_{k=0}^{n} \binom{n+1}{k} \left[\sum_{\ell \geq 1} \mathbb{E}\left[\left(\sum_{j=1}^{\ell} \mathbf{1}_{\{B_j > t-s\}}\right)^{n+1-k}\right] \mathbf{K}_\ell(s)\right] \mathbf{M}_{b,k}(s). \tag{9}$$

Readers should note as well that analogous results can also be derived for shot-noise models fed by nonhomogeneous MAPs: we leave this to the interested reader.

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