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A Conditional-Logic Interpretation for Miller-Tucker-Zemlin Inequalities and Extensions

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Abstract Routing problems that seek to traverse a set of cities are faced with the challenge of avoiding subtours. To address this challenge, attention has been given to devising subtour elimination constraints. A classical approach is through the use of "Miller-Tucker-Zemlin" (MTZ) inequalities. MTZ inequalities have the advantage of being few in number but have the disadvantage of yielding weak continuous relaxations. As a result, strengthenings have been computed over time in a seemingly unrelated fashion. In this note, we provide a unifying conditional-logic interpretation of MTZ inequalities for the Traveling Salesman Problem (TSP). Our emphasis is on *linear* inequalities but our analysis also provides a new family of tightened *quadratic* forms. We apply the interpretation to the more general Capacitated Vehicle Routing Problem (CVRP), both explaining existing and motivating new inequalities.

Keywords Miller-Tucker-Zemlin Inequalities · Conditional Logic · Traveling Salesman · Vehicle Routing

1 Introduction

The Traveling Salesman Problem (TSP) is a classical NP-hard routing problem that has been well-studied in the literature. Given a base city and a collection of $(n-1)$ additional cities, the TSP seeks a minimum-cost tour that starts and ends at the base city and visits every other city exactly once. Various exact and heuristic methods have been suggested to solve the TSP; prevalent amongst these methods is (mixed) 0-1 linear programming. When using a mixed 0-1 linear form, a key computational challenge is the efficient elimination of subtours. Different families of subtour elimination constraints are available in the literature to address this challenge, with an important family being the "Miller-Tucker-Zemlin" (MTZ) inequalities.

The MTZ inequalities were introduced in [14]. Compared with other subtour elimination constraints such as the DFJ inequalities [4], they have the advantage of being few in number. However, they are also known to suffer from weak continuous relaxations. Various strengthenings of the MTZ inequalities can be found in the literature [1,5,8,9,16], with the works of [8,9] in a lifted variable space. A summary is in [2]. Interestingly, the strengthened MTZ inequalities have been derived over time using seemingly unrelated approaches.

In this note, we provide a conditional-logic interpretation that serves to unify the MTZ inequalities [14] and their strengthenings, and to also afford new inequalities, both linear and quadratic. Of significance

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is that the new inequalities are in the same variable space as the original MTZ forms. The concept of conditional logic, which was first introduced in [15] and later studied in [13], is generally applicable to mixed 0-1 programs. However, there is a great deal of flexibility in the application, and effective implementations rely on the exploitation of problem structure to strategically form logical implications. No consideration in [13,15] was given to MTZ inequalities.

When applied to the TSP and CVRP within this paper, conditional-logic consists of two steps. The first step computes valid nonlinear inequalities in the spirit of the reformulation-linearization technique [17] by multiplying given restrictions with functions of binary variables. Here, nonlinear inequalities are obtained by computing the product of a binary expression with a linear restriction that is not necessarily valid for the motivating problem, but is "conditionally" valid when the binary expression realizes the value 1. The inequalities are strategically computed so that, in the second step, they can be surrogated to yield linear restrictions. The second step is the surrogation. For our purposes, we adopt the terminology that "conditional logic" refers to computations in which the multiplying binary expressions are linear, and that "compound conditional logic" refers to computations in which the multiplying binary expressions are nonlinear (quadratic).

Our contributions to the TSP primarily focus on the derivation of *linear* inequalities but, in the process, we also generate *quadratic* restrictions. We begin by showing that conditional logic readily gives the tightened MTZ inequalities of [5]. We then show how compound conditional logic gives quadratic inequalities which are tighter than those of [16], and how relaxations of these quadratic inequalities motivate various linear families of [1] that were originally discovered using PORTA [3].

The conditional-logic interpretation for the TSP is generally extendable to other routing problems that seek to eliminate subtours via MTZ inequalities. Included here is a generalization of the TSP known as the Capacitated Vehicle Routing Problem (CVRP). For the CVRP, we show how conditional logic motivates restrictions that were collectively provided by [5], [10], [11], and [12]. We then use compound conditional logic to compute new families of inequalities which generalize those of [1] for the TSP. This generalization is possible despite the fact that the computational approach of [1] does not extend to the CVRP due to problem structure.

This paper is organized as follows. Sections 2 and 3 focus on the TSP and CVRP, respectively. Each section is divided into three subsections, with the first reviewing a mathematical formulation that uses MTZ inequalities, the second applying conditional logic to obtain strengthened known inequalities, and the third invoking compound conditional logic to compute new, tightened restrictions. Section 4 provides a summary and concluding remarks.

2 Traveling Salesman Problem

The TSP is a classical problem that, given a collection of n cities, seeks a minimal-cost permutation where, for each pair of distinct cities i and j, a cost is incurred for having city i immediately precede city j. The last city in a permutation is considered to immediately precede the first. We assume without loss of generality that city 1 is the "base" which occurs first in the permutation. We also assume that $n \geq 6$. The interpretation is that of a salesman who, starting and ending at the base city 1, seeks a minimal cost circuit through the cities, with a cost incurred for traveling between each pair of cities i and j. As the cost for city i immediately preceding city j is not restricted to be the cost of city j immediately preceding city i, we consider the asymmetric TSP. The first subsection below gives a formulation using MTZ inequalities, the second subsection shows how conditional logic readily generates known, tightened inequalities, and the third subsection uses compound conditional logic both to generate a new family of quadratic inequalities and to theoretically explain known linear inequalities that were previously discovered computationally.

2.1 Mathematical Formulation

Given any two distinct cities i and j, let c_{ij} denote the cost for having city i immediately precede city j in the circuit. Define $2\binom{n}{2}$ binary variables $\boldsymbol{x}$ so that

$$x_{ij} = \begin{cases} 1 & \text{if city } i \text{ immediately precedes city } j \\ 0 & \text{otherwise} \end{cases} \quad \forall\ (i,j),\ i \neq j.$$

Also define $(n-1)$ variables $\boldsymbol{u} \equiv (u_2, u_3, \ldots, u_n)$ having the interpretation that $u_j = k$ indicates that city j is located in the k^{th} position after the first. Then a formulation is as follows.

$$\overline{\text{TSP}}: \text{minimize} \sum_i \sum_{j \neq i} c_{ij} x_{ij}$$

subject to

$$u_j - u_i \geq (2-n) + (n-1)x_{ij} + (n-3)x_{ji} \quad \forall\, (i,j),\ i,j \geq 2,\ i \neq j \tag{1}$$

$$2 - x_{1j} + (n-3)x_{j1} \leq u_j \leq (n-2) + (3-n)x_{1j} + x_{j1} \quad \forall\, j \geq 2 \tag{2}$$

$$\sum_{j \neq i} x_{ij} = 1 \quad \forall\, i \tag{3}$$

$$\sum_{i \neq j} x_{ij} = 1 \quad \forall\, j \tag{4}$$

$\boldsymbol{x}$ binary

The constraints of $\overline{\text{TSP}}$ operate as follows. Inequalities (1) are a strengthened version, due to [5], of the $(n-1)(n-2)$ MTZ inequalities of [14] that take the form

$$u_j - u_i \geq (2-n) + (n-1)x_{ij} \quad \forall\, (i,j),\ i,j \geq 2,\ i \neq j \tag{5}$$

and serve to eliminate subtours. Given the earlier-stated interpretation of the variables u_j, inequalities (2), also due to [5], are tightened versions of the inequalities

$$1 \leq u_j \leq n-1 \quad \forall\, j \geq 2 \tag{6}$$

that follow from [14]. Equations (3) enforce that each city i has a single city immediately following it, while equations (4) enforce that each city j has a single city immediately preceding it.

2.2 Conditional Logic for the TSP

We begin our study on conditional logic by showing how our simplest application yields inequalities (1) and (2) of [5].

To obtain (1), for each $(i,j),\ i,j \geq 2,\ i \neq j$, make the following conditional statements based on logic (upon using the above interpretation that $u_j = k$ indicates city j is located in the k^{th} position after the first).

$$\begin{array}{ll} \text{If } x_{ij} = 1, & \text{then } u_j - u_i = 1. \\ \text{If } x_{ji} = 1, & \text{then } u_j - u_i = -1. \\ \text{If } 1 - x_{ij} - x_{ji} = 1, & \text{then } u_j - u_i \geq 2 - n. \end{array}$$

Each of the expressions x_{ij}, x_{ji}, and $(1 - x_{ij} - x_{ji})$ found in the antecedents of these statements is binary for all feasible solutions to $\overline{\text{TSP}}$. Consequently, we can multiply each such expression by its associated consequence to obtain the following three quadratic restrictions that are valid for every $(i,j),\ i,j \geq 2, i \neq j$.

$$\begin{aligned} x_{ij}(u_j - u_i) &= x_{ij}, \\ x_{ji}(u_j - u_i) &= -x_{ji}, \\ (1 - x_{ij} - x_{ji})(u_j - u_i) &\geq (2-n)(1 - x_{ij} - x_{ji}). \end{aligned}$$

The validity holds true for the following reason. For each restriction, if the associated antecedent is 0, then the result is trivial. Otherwise, the antecedent is 1 and the result holds by the associated conditional statement. We sum these restrictions to obtain

$$u_j - u_i \geq x_{ij} - x_{ji} + (2-n)(1 - x_{ij} - x_{ji}),$$

which is (1).

Observe the correspondence between (1) and the above conditional-logic statements. If $x_{ij} = 1$, then x_{ji} must equal to 0 so that (1) gives $u_j - u_i \geq 1$, which is a relaxation of the first logical consequence. Alternately, if $x_{ji} = 1$, then x_{ij} must equal to 0 so that (1) gives $u_j - u_i \geq -1$, which is a relaxation of the second logical consequence. Finally, if $x_{ij} = x_{ji} = 0$, then (1) gives $u_j - u_i \geq 2 - n$, which is the third logical consequence.

To obtain (2), for each $j \geq 2$, make the following conditional statements.

$$\begin{array}{ll} \text{If } x_{1j} = 1, & \text{then } u_j = 1. \\ \text{If } x_{j1} = 1, & \text{then } u_j = n-1. \\ \text{If } 1 - x_{1j} - x_{j1} = 1, & \text{then } 2 \leq u_j \leq n-2. \end{array}$$

Each of the expressions x_{1j}, x_{j1}, and $(1 - x_{1j} - x_{j1})$ found in the antecedents of these statements is binary for all feasible solutions to $\overline{\text{TSP}}$. Then, upon separately considering the two inequalities in the third statement, we multiply each such expression by its consequence and sum as above to obtain (2). Here, $2 \leq u_j$ and $u_j \leq n-2$ of the third statement give rise to the left and right inequalities, respectively.

A key attribute of the above two conditional-logic applications is that the functions found in the antecedents sum to unity. This sum allowed all the quadratic terms in the variables $\boldsymbol{u}$ to vanish. For these examples, the antecedents are linear functions. However, as pointed out in [13], the antecedents can be nonlinear to represent conjunctions of expressions of binary variables, leading to "compound conditional logic" statements. Within the context of MTZ inequalities for the TSP, this logic provides insights into, explanations for, and extensions of, published works as discussed in the following subsection.

2.3 Compound Conditional Logic for the TSP

Richer families of MTZ inequalities than (1) and (2) can be obtained for the TSP using compound conditional logic statements that employ quadratic expressions of the binary variables as the antecedents. A paper [16] gives a family of *quadratic* MTZ inequalities, and a paper [1] reports various families of *linear* MTZ inequalities. Our approach generates a family of quadratic MTZ inequalities that are tighter than those of [16], and that motivates the inequalities of [1] through appropriate weakenings and surrogates.

Given an instance of the TSP, consider any $(p, q),\ p, q \geq 2, p \neq q$, and define the set S_{pq} to consist of three functions of the variables x_{pq} and x_{qp} as

$$S_{pq} \equiv \{x_{pq}, x_{qp}, 1 - x_{pq} - x_{qp}\}\,.$$

Now, consider any distinct $(i, j, k), i, j, k \geq 2$, and the associated sets S_{ij}, S_{jk}, and S_{ik}. Suppose that, for each of the 27 possible different ways in which exactly one function can be selected from each of the three sets, we compute a cubic "product function" by multiplying the three selections together. Each of these 27 product functions is binary, and they collectively sum to unity. As a result, we can use these product functions as antecedents of conditional statements, and the summation process will eliminate all quadratic expressions in the variables $\boldsymbol{u}$.

Recognizing that $x_{pq}x_{rq} = x_{pq}x_{pr} = x_{pq}x_{qr}x_{rp} = 0$ for all such (p, q, r), we have that 14 of the 27 product functions must equal to 0 since then $x_{pq}(1 - x_{qr} - x_{rp}) = x_{pq}(1 - x_{qr} - x_{rq})(1 - x_{pr} - x_{rp})$ and $x_{rp}x_{pq} = x_{rp}x_{pq}(1 - x_{qr} - x_{rq})$. The remaining 13 product functions are found in the second column of Table 1 labeled "Antecedent." The first column gives the conditional logic statement number # as "CL#," and the third column gives the associated consequence. More specifically, the number in the third column represents the value of $*$ in the expression $u_j - u_i \geq *$ so that, for example, the consequence of statement CL1 having the antecedent $x_{ij}x_{jk} = 1$ is $u_j - u_i \geq 1$. For ease of discussion, we list all consequences as "greater-than-or-equal to," though the consequences for statements CL1 through CL7 and CL9 hold with equality. Within the table, $\tau = 1 - x_{ij} - x_{ji} - x_{jk} - x_{kj} - x_{ik} - x_{ki} + x_{ij}x_{jk} + x_{ik}x_{kj} + x_{ji}x_{ik} + x_{jk}x_{ki} + x_{ki}x_{ij} + x_{kj}x_{ji}$. (It is easily seen that each of the 13 expressions in the antecedents is binary, e.g. see [7]. Also, our earlier assumption that $n \geq 6$ ensures that each of the antecedents can realize value 1. If, for example, $n = 5$, then the antecedent for CL13 must realize value 0 and need not be involved in the conditional logic process.)

As with the conditional-logic arguments of the previous subsection, we constructively compute valid inequalities, this time quadratic, by taking products of the antecedents with the consequences. In particular, we first multiply the quadratic function found in each antecedent by its consequence to obtain the 13 valid cubic inequalities given by

$$\begin{aligned} x_{ij}x_{jk}(u_j - u_i) &\geq x_{ij}x_{jk} \\ x_{ik}x_{kj}(u_j - u_i) &\geq 2x_{ik}x_{kj} \\ &\vdots \\ \tau(u_j - u_i) &\geq (2 - n)\tau. \end{aligned}$$

Table 1: Conditional Logic Statements for TSP

No.	**Antecedent** (If function $= 1$)	**Consequence** ($u_j - u_i \geq *$)	**Weakened Consequence** ($u_j - u_i \geq *$) WC1	WC2	WC3	WC4	WC5	WC6
CL1	$x_{ij}x_{jk}$	1						
CL2	$x_{ik}x_{kj}$	2	$2-n$			$6-2n$		
CL3	$x_{ji}x_{ik}$	-1						
CL4	$x_{jk}x_{ki}$	-2	$2-n$		$4-2n$			
CL5	$x_{ki}x_{ij}$	1						
CL6	$x_{kj}x_{ji}$	-1						
CL7	$x_{ij}(1-x_{jk}-x_{ki})$	1		$4-n$		$5-n$		
CL8	$x_{ik}(1-x_{kj}-x_{ji})$	$3-n$	$2-n$			$6-2n$	$7-2n$	
CL9	$x_{ji}(1-x_{ik}-x_{kj})$	-1		$-n$	$-n$			
CL10	$x_{jk}(1-x_{ki}-x_{ij})$	$2-n$		$1-n$	$4-2n$			$3-2n$
CL11	$x_{ki}(1-x_{ij}-x_{jk})$	$2-n$		$1-n$	$4-2n$		$3-2n$	
CL12	$x_{kj}(1-x_{ji}-x_{ik})$	$3-n$	$2-n$			$6-2n$		$7-2n$
CL13	τ	$2-n$		$4-2n$	$4-2n$	$6-2n$	$5-2n$	$5-2n$

We then sum the 13 inequalities. Upon performing these operations for every distinct $(i,j,k),\ i,j,k \geq 2$, we obtain the family of quadratic inequalities

$$\begin{aligned} u_j - u_i \geq (2-n) + (n-1)x_{ij} + (n-3)x_{ji} + (x_{ik} + x_{kj}) - (x_{ji}x_{ik} + x_{kj}x_{ji}) \\ + (n-2)x_{ik}x_{kj} + (n-4)x_{jk}x_{ki} \ \ \forall \text{ distinct } (i,j,k),\ i,j,k \geq 2. \end{aligned} \tag{7}$$

Here, as with all previous applications of conditional-logic, and as mentioned above, since the sum of the functions found in the antecedents is unity, all nonlinear terms involving the variables $\boldsymbol{u}$ vanish.

Inequalities (7) are of theoretical importance because they explain known quadratic and linear inequalities over three cities for the TSP. In fact, weakened conditional-logic statements obtained by lessening the consequences of column 3 motivate the explanations. As a first example, for each distinct $(i,j,k), i,j,k \geq 2$, inequality (7) is tighter than the quadratic inequality of [16] given by

$$\begin{aligned} u_j - u_i \geq (2-n) + (n-1)x_{ij} + (x_{ik} + x_{kj}) + (n-2)x_{ik}x_{kj} \\ + (n-4)(x_{ji}x_{ik} + x_{kj}x_{ji} + x_{jk}x_{ki}), \end{aligned} \tag{8}$$

in that the right side of (7) exceeds that of (8) by the nonnegative quantity $(n-3)x_{ji}(1-x_{ik}-x_{kj})$. In terms of conditional logic, inequality (8) can be obtained by weakening the consequence of statement CL9 from -1 to $(2-n)$.

Columns 4 through 9 of Table 1 record weakened consequences of column 3 that lead to known linear inequalities, with each column corresponding to a different family. To explain, consider column 4 that corresponds to weakened consequence 1, denoted "WC1." No entry in a given row within this column indicates that the consequence found in column 3 remains unchanged while a numeric entry denotes that the consequence of column 3 is decreased to the given value. Thus, in computing the inequality corresponding to WC1, for example, the consequences of statements CL2, CL4, CL8, and CL12 are each decreased to the value $(2-n)$ and all other consequences are left unchanged from column 3. The resulting inequality is (1) of $\overline{\text{TSP}}$, giving us a different conditional-logic approach for obtaining (1) than that described in Subsection 2.2.

Weakened consequences WC2 through WC6 result in each of five different families of inequalities of [1] as described below. These families were discovered using PORTA [3].

(i) Weakened consequence WC2 results in the "clique inequality"

$$u_j - u_i \geq (4-2n) + (n-1)(x_{ik} + x_{kj}) + nx_{ij} + (n-3)(x_{jk} + x_{ki}) + (n-4)x_{ji}. \tag{9}$$

(ii) Weakened consequence WC3 results in the "2PATH inequality"

$$u_j - u_i \geq (4-2n) + (2n-3)x_{ij} + (n-4)x_{ji} + (n-1)(x_{ik} + x_{kj}). \tag{10}$$

(iii) Weakened consequence WC4 results in the "2PATH inequality"

$$u_j - u_i \geq (6-2n) + (n-1)x_{ij} + (2n-7)x_{ji} + (n-4)(x_{jk} + x_{ki}). \tag{11}$$

Table 2: Summary of the MTZ inequalities for the TSP

Valid inequalities	Reference
(5), (6)	Miller, Tucker and Zemlin (1960) [14]
(1), (2)	Desrochers and Laporte (1991) [5]
(8)	Sherali and Driscoll (2002) [16]
(9)-(11), (12), (14)	Bektaş and Gouveia (2014) [1]
(7)	this paper

(iv) Weakened consequence WC5 results in the inequality

$$-2u_i + u_j + u_k \geq (10-4n) + (2n-2)(x_{ij}+x_{ik}) + (2n-8)(x_{ji}+x_{ki}) + (2n-5)(x_{jk}+x_{kj}) \quad (12)$$

as follows. The values of consequence WC5 of Table 1 give the quadratic inequality

$$u_j - u_i \geq (5-2n) + (2n-4)x_{ij} + (2n-6)x_{ji} + 2x_{ik} - 2x_{ki} + (n-3)x_{jk} + (n-2)x_{kj} + T^1_{ijk}, \quad (13)$$

where $T^1_{ijk} \equiv (n-3)(x_{ik}x_{kj} - x_{ij}x_{jk}) + (n-2)(x_{jk}x_{ki} - x_{kj}x_{ji}) + 2(x_{ki}x_{ij} - x_{ji}x_{ik})$. Interchange the indices j and k within (13) to obtain a valid inequality for $u_k - u_i$ and add this inequality to (13) to obtain (12), upon using that $T^1_{ijk} + T^1_{ikj} = 0$.

(v) Weakened consequence WC6 results in the inequality

$$-u_i + 2u_j - u_k \geq (10-4n) + (2n-8)(x_{ji}+x_{jk}) + (2n-2)(x_{ij}+x_{kj}) + (2n-5)(x_{ik}+x_{ki}). \quad (14)$$

as follows. The values of consequence WC6 of Table 1 give the quadratic inequality

$$u_j - u_i \geq (5-2n) + (2n-4)x_{ij} + (2n-6)x_{ji} + (n-2)x_{ik} + (n-3)x_{ki} - 2x_{jk} + 2x_{kj} + T^2_{ijk}, \quad (15)$$

where $T^2_{ijk} \equiv 2(x_{ij}x_{jk} - x_{kj}x_{ji}) + (n-3)(x_{ik}x_{kj} - x_{ki}x_{ij}) + (n-2)(x_{jk}x_{ki} - x_{ji}x_{ik})$. Interchange the indices i and k within (15) to get a valid inequality in terms of $u_j - u_k$ and add this inequality to (15) to obtain (14), upon using that $T^2_{ijk} + T^2_{kji} = 0$.

Thus, five different families of inequalities found in [1] are available via conditional logic. The logic allows us, in the next section, to generate new MTZ inequalities for the CVRP.

We summarize the origins of the MTZ inequalities for the TSP in Table 2.

3 Capacitated Vehicle Routing Problem

The single-depot CVRP is a generalization of the TSP. It seeks to route at most m vehicles amongst n cities so that each city is visited exactly once, so that each route begins and ends at base city 1 with no subtours, and so that all city demands for a product are met. Each of the m vehicles has the same capacity Q for the product and each city j has some demand $q_j > 0$. As with the TSP, there is a cost for having city i immediately precede city j in a circuit, and the objective is to minimize the overall cost while meeting the demands of all cities.

3.1 Mathematical Formulation

Different mathematical formulations exist for the CVRP (see [18] for example). A strengthened form that uses modified MTZ inequalities in the same binary variables $\boldsymbol{x}$ and continuous variables $\boldsymbol{u}$ as $\overline{\text{TSP}}$, and uses the same objective function coefficients c_{ij}, is given below. The variables $\boldsymbol{u}$, in this context, have the interpretation that u_j for $j \geq 2$ is the amount of vehicle capacity that has been collectively allocated to the cities on the tour from city 1 through city j.

$\overline{\text{CVRP}}$: minimize $\sum_i \sum_{j \neq i} c_{ij}x_{ij}$

subject to

$$u_j - u_i \geq (q_j - Q) + Qx_{ij} + (Q - q_i - q_j)x_{ji} \qquad \forall\,(i,j),\ i,j \geq 2,\ i \neq j \tag{16}$$

$$q_j + \sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ij} \leq u_j \leq Q - (Q - \max_{\substack{i \geq 2 \\ i \neq j}}\{q_i\} - q_j)x_{1j} - \sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji} \qquad \forall\, j \geq 2 \tag{17}$$

$$u_j \leq q_j x_{1j} + Q(1 - x_{1j}) \qquad \forall\, j \geq 2 \tag{18}$$

$$\sum_{j \geq 2} x_{1j} \leq m \tag{19}$$

$$\sum_{i \geq 2} x_{i1} \leq m \tag{20}$$

$$\sum_{j \neq i} x_{ij} = 1 \qquad \forall\, i \geq 2 \tag{21}$$

$$\sum_{i \neq j} x_{ij} = 1 \qquad \forall\, j \geq 2 \tag{22}$$

$\boldsymbol{x}$ binary

The constraints of $\overline{\text{CVRP}}$ perform as follows. Inequalities (16), due to [5] as corrected by [11], are a strengthened version of the inequalities

$$u_j - u_i \geq (q_j - Q) + Qx_{ij} \quad \forall\,(i,j),\ i,j \geq 2,\ i \neq j, \tag{23}$$

that were used by [12] to eliminate subtours. Inequalities (17), also due to [5] with the right inequalities as corrected by [11], are a strengthened version of the standard inequalities

$$q_j \leq u_j \leq Q \quad \forall\, j \geq 2, \tag{24}$$

that follow from [12]. Inequalities (18), due to [10], are used to restrict $\boldsymbol{u}$. Restrictions (19)–(22) are a variant of (3) and (4) that allow for up to m vehicles to enter/leave base city 1.

The structure of $\overline{\text{CVRP}}$ allows for certain assumptions. First, we assume without loss of generality that $q_j \leq Q$ for all $j \geq 2$ since otherwise the problem is infeasible. Second, and as noted by [11], for each (i,j), $2 \leq i < j \leq n$, having $q_i + q_j > Q$, it is not possible for a vehicle to travel directly between cities i and j so that we can set $x_{ij} = x_{ji} = 0$. Hence, we assume that only those variables x_{ij} having $q_i + q_j \leq Q$ are found within $\overline{\text{CVRP}}$ and that summations are taken over only those (i,j) pairs having $q_i + q_j \leq Q$. Third, we consequently consider only those inequalities of (16) having (i,j) with $q_i + q_j \leq Q$ since $x_{ij} = x_{ji} = 0$ reduces (16) to $u_j - u_i \geq q_j - Q$ which is implied by the consequence of (17) that $q_j \leq u_j$ and $u_i \leq Q$.

For the special case of the CVRP wherein each city has demand $q_j = 1$ and there exists a single vehicle $m = 1$ having capacity $Q = n - 1$, the CVRP simplifies to the TSP. For such q_j and Q, (16) reduces to (1) while (17) reduces, by (21) and (22), to

$$2 - x_{1j} \leq u_j \leq (n-2) + (3-n)x_{1j} + x_{j1} \ \ \forall\, j \geq 2, \tag{25}$$

which is a weakened form of (2) since, for each $j \geq 2$, the left side of (2) exceeds that of (25) by the nonnegative quantity $(n-3)x_{j1}$. (The tightened form (2) for the TSP relies on $m = 1$.) Also, inequalities (18) reduce to

$$u_j \leq (n-1) + (2-n)x_{1j} \ \ \forall\, j \geq 2. \tag{26}$$

The right side of (26) can be obtained by adding the quantity $1 - x_{1j} - x_{j1}$ to the right side of (2). Thus, inequalities (26) cannot help the TSP in the presence of (2) because the TSP requires that $x_{1j} + x_{j1} \leq 1$, but they can help the CVRP since it is possible in the CVRP for a vehicle to visit a single city j so that $x_{1j} = x_{j1} = 1$.

3.2 Conditional Logic for the CVRP

While the approach of [1] that uses PORTA [3] to discover new inequalities is not transferable from the TSP because the values of q_j can vary from one instance of the CVRP to another, conditional logic is transferable and allows for the computation of (16)–(18) as well as new families of inequalities which subsume those for the TSP. We begin by considering (16)–(18). Thereafter, in Subsection 3.3, we use compound conditional logic to obtain our new families.

To obtain (16), and paralleling the logic of Subsection 2.2, we construct logical statements, but with the consequences adjusted to reflect the city demands q_j and vehicle capacity Q of the CVRP. We have the following conditional statements associated with each $(i, j),\ i, j \geq 2,\ i \neq j$, where, as noted above, $q_i + q_j \leq Q$.

$$\begin{array}{ll} \text{If } x_{ij} = 1, & \text{then } u_j - u_i = q_j. \\ \text{If } x_{ji} = 1, & \text{then } u_j - u_i = -q_i. \\ \text{If } 1 - x_{ij} - x_{ji} = 1, & \text{then } u_j - u_i \geq q_j - Q. \end{array}$$

Each of the expressions x_{ij}, x_{ji}, and $1 - x_{ij} - x_{ji}$ found in the antecedents of these statements is binary for all feasible solutions to $\overline{\text{CVRP}}$, and they sum to unity. Multiply each such expression by its associated consequence and sum to obtain (16).

Relative to (17) and (18), we have the following conditional statements associated with each $j \geq 2$.

$$\begin{array}{ll} \text{If } x_{1j} = 1, & \text{then } u_j = q_j. \\ \text{If } (1 - x_{1j})x_{j1} = 1, & \text{then } q_j + \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ij} \leq u_j \leq Q. \\ \text{If } (1 - x_{1j})(1 - x_{j1}) = 1, & \text{then } q_j + \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ij} \leq u_j \leq Q - \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji}. \end{array}$$

Unlike the computation of (2) via conditional logic for the TSP, we include quadratic expressions in two of the above antecedents because it is possible to have a vehicle visit only a single city j so that $x_{1j} = x_{j1} = 1$. Each of the expressions x_{1j}, $(1 - x_{1j})x_{j1}$, and $(1 - x_{1j})(1 - x_{j1})$ found in the antecedents of these statements is binary for all feasible solutions to $\overline{\text{CVRP}}$, and they sum to unity. Then, upon separately considering the two inequalities in the second and third statements, we multiply each such expression by its associated consequence and sum to obtain

$$q_j + \sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ij} \leq u_j \leq q_j x_{1j} + (x_{1j} - 1)(\sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji} - Q) \ \ \forall\, j \geq 2, \tag{27}$$

where we have used that $x_{1j} \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ij} = x_{j1} \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji} = 0$ for all $j \geq 2$.

For each $j \geq 2$, the left inequalities of (17) and (27) are the same. The right inequality of (27) is quadratic since it contains the expression $x_{1j} \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji}$, but it can be weakened to yield the right inequality of (17) and inequality (18). To see this, $x_{1j} \sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji}$ is bounded above by each of $x_{1j} \max\limits_{\substack{i \geq 2 \\ i \neq j}}\{q_i\}$ and $\sum\limits_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji}$. Substituting the first upper bound into (27) gives the right inequality of (17) and substituting the second upper bound gives (18).

3.3 Compound Conditional Logic for the CVRP

Following Subsection 2.3 for the TSP, we generate strengthened inequalities for the CVRP. We consider city triples $(i, j, k), i, j, k \geq 2$, for which $q_i + q_j + q_k \leq Q$ so that the vehicle has sufficient capacity Q to visit all three cities i, j, and k before returning to the base city. Our generalized inequalities are not exhaustive, and it is beyond the scope of this paper to enumerate all possibilities. However, we clearly show how conditional logic provides the fundamental machinery for making generalizations, subsuming (9)–(12) and (14) in the process.

We form the same type of 13 conditional logic statements as used for the TSP in Table 1, but adjust the consequences to reflect the city demands q_j and vehicle capacity Q. Consider Table 3. In a similar manner to the presentation of Table 1, columns 2 and 3 of Table 3 give the antecedents and consequences, respectively, for each of the 13 conditional-logic statements numbered CL1 through CL13 in column 1.

Table 3: Conditional Logic Statements for CVRP

No.	**Antecedent** (If function = 1)	**Consequence** ($u_j - u_i \geq *$)	**Weakened Consequence** ($u_j - u_i \geq *$) WC1	WC2	WC3
CL1	$x_{ij}x_{jk}$	q_j			
CL2	$x_{ik}x_{kj}$	$q_j + q_k$			$q_i + 2q_j + q_k - 2Q$
CL3	$x_{ji}x_{ik}$	$-q_i$			
CL4	$x_{jk}x_{ki}$	$-q_i - q_k$		$q_j + q_k - 2Q$	
CL5	$x_{ki}x_{ij}$	q_j			
CL6	$x_{kj}x_{ji}$	$-q_i$			
CL7	$x_{ij}(1 - x_{jk} - x_{ki})$	q_j	$\frac{q_i+3q_j}{2} + q_k - Q$		$q_i + 2q_j + q_k - Q$
CL8	$x_{ik}(1 - x_{kj} - x_{ji})$	$q_j + q_k - Q$			$q_i + 2q_j + q_k - 2Q$
CL9	$x_{ji}(1 - x_{ik} - x_{kj})$	$-q_i$	$-q_i - Q$	$-q_i - Q$	
CL10	$x_{jk}(1 - x_{ki} - x_{ij})$	$q_j - Q$	$\frac{q_j-q_i}{2} - Q$	$q_j + q_k - 2Q$	
CL11	$x_{ki}(1 - x_{ij} - x_{jk})$	$q_j - Q$	$\frac{q_j-q_i}{2} - Q$	$q_j + q_k - 2Q$	
CL12	$x_{kj}(1 - x_{ji} - x_{ik})$	$q_j + q_k - Q$			$q_i + 2q_j + q_k - 2Q$
CL13	τ	$q_j - Q$	$q_j + q_k - 2Q$	$q_j + q_k - 2Q$	$q_i + 2q_j + q_k - 2Q$

As before, we multiply the quadratic function found in each antecedent by its consequence and sum to obtain

$$\begin{aligned} u_j - u_i \geq\ & (q_j - Q) + Qx_{ij} + (Q - q_i - q_j)x_{ji} + q_k(x_{ik} + x_{kj}) \\ & - q_k(x_{ji}x_{ik} + x_{kj}x_{ji}) + (Q - q_k)x_{ik}x_{kj} \\ & + (Q - q_i - q_j - q_k)x_{jk}x_{ki} \ \ \forall \text{ distinct } (i, j, k),\ i, j, k \geq 2. \end{aligned} \tag{28}$$

Inequalities (28) generalize (7) in that by fixing $q_i = q_j = q_k = 1$ and $Q = n - 1$ in the latter, we obtain the former. Similar to (7), the validity of (28) is constructive since its computation consists of the summing of 13 valid cubic inequalities.

Columns 4 through 6 of Table 3 record weakened consequences of column 3 that lead to new families of linear inequalities. As with columns 4 through 9 of Table 1, no entry in some row for columns 4 through 6 of Table 3 indicates that the consequence found in column 3 remains unchanged. The resulting inequalities found below generalize (9)–(11) for the TSP.

(i) Weakened consequence WC1 results in the inequality

$$\begin{aligned} u_j - u_i \geq\ & (q_j + q_k - 2Q) + Q(x_{ik} + x_{kj}) + \left(Q + \frac{q_i + q_j}{2}\right) x_{ij} \\ & + \left(Q - q_k - \frac{q_i + q_j}{2}\right)(x_{jk} + x_{ki}) + (Q - q_i - q_j - q_k)x_{ji}. \end{aligned} \tag{29}$$

(ii) Weakened consequence WC2 results in the inequality

$$u_j - u_i \geq (q_j + q_k - 2Q) + (2Q - q_k)x_{ij} + (Q - q_i - q_j - q_k)x_{ji} + Q(x_{ik} + x_{kj}). \tag{30}$$

(iii) Weakened consequence WC3 results in the inequality

$$\begin{aligned} u_j - u_i \geq\ & (q_i + 2q_j + q_k - 2Q) + Qx_{ij} + (2Q - 2q_i - 2q_j - q_k)x_{ji} \\ & + (Q - q_i - q_j - q_k)(x_{jk} + x_{ki}). \end{aligned} \tag{31}$$

Inequalities (29)–(31) reduce to (9)–(11), respectively, when $q_i = q_j = q_k = 1$ and $Q = n - 1$.

Inequalities (12) and (14) also generalize to the CVRP. However, unlike the TSP which has $q_i = q_j = q_k = 1$ and unlike the computation of (29)–(31), the derivations must address the relative magnitudes of the demands q_i, q_j, and q_k to ensure that the conditional-logic consequences remain valid. The reader is referred to the Appendix for such generalizations and their derivations.

We summarize the origins of the MTZ inequalities for the CVRP in Table 4. Included in this summary are inequalities (32), (34), (36), and (38) that are derived in the Appendix.

Below is a small numeric example showing that our new cuts of this subsection can eliminate feasible points from the continuous relaxation of $\overline{\text{CVRP}}$.

Table 4: Summary of the MTZ inequalities for the CVRP

Valid Inequalities	Reference
(23), (24)	Kulkarni and Bhave (1985) [12]
(16), (17)	Desrochers and Laporte (1991) [5]
	and Kara, Laporte and Bektaş (2004) [11]
(18)	Kara (2010) [10]
(28)-(31), (32), (34), (36), (38)	this paper

Example 1 Consider an instance of the CVRP with $n = 6$ cities that each have demand $q_j = 1$ and at most $m = 2$ vehicles having capacity $Q = 4$. The solution $(\boldsymbol{u}, \boldsymbol{x})$ with $(u_2, u_3, u_4, u_5, u_6) = (2, 2, 2, 1, 1)$ and $\boldsymbol{x}$ having $x_{15} = x_{16} = x_{51} = x_{61} = 1$, $x_{23} = x_{24} = x_{32} = x_{34} = x_{42} = x_{43} = \frac{1}{2}$, and all twenty remaining $x_{ij} = 0$ is feasible to the continuous relaxation of $\overline{\text{CVRP}}$. However, this point violates each of (29), (30), and (31) for $(i, j, k) = (2, 3, 4)$, giving $0 \geq 3$, $0 \geq 2$, and $0 \geq \frac{1}{2}$, respectively.

We close this subsection by mentioning that the CVRP can be modified to reflect each vehicle having a limitation measured in terms of factors other than product capacity, such as travel distance or transportation cost. For these modifications, the variables u_j will again represent the amount of the vehicle limitation that has been collectively allocated to the cities on the tour from city 1 through city j, and Q will again represent the vehicle limitation. The above analysis to obtain strengthened inequalities then follows in a similar manner.

4 Conclusions

This note presents a conditional-logic framework for deriving and tightening MTZ-type inequalities that are used to eliminate subtours in the TSP and CVRP. The framework consists of the two steps of conditional logic and surrogation, with the logic strategically applied so that a surrogation of the resulting nonlinear inequalities provides linear restrictions. Depending on the application, both linear and nonlinear restrictions can be computed. We showed how tightened MTZ inequalities of [5] result from a simple application of conditional logic. Compound conditional logic gives quadratic inequalities that are tighter than those of [16], and relaxations thereof explain various inequalities of [1] that were originally discovered computationally. For the CVRP, we showed how conditional logic yields inequalities provided by [5], [10], [11], and [12]. We also derived new families of inequalities. Notably, while the computational analysis of [1] to discover new inequalities for the TSP is not extendable to the CVRP due to the individual city demands present in the more general CVRP, conditional logic provides the machinery for computing CVRP inequalities that generalize those of [1] in the sense that, when there is only a single vehicle in the CVRP with sufficient capacity to visit all cities, the new inequalities reduce to the form of [1].

Finally, we mention that while this paper focuses on the TSP and CVRP, the same conditional logic framework is applicable to other problems that seek to traverse a set of cities while avoiding subtours. As with the TSP and CVRP, the framework leads to known and tightened MTZ inequalities. Such problems include the Linear Ordering Problem, Target Visitation Problem, and Quadratic Traveling Salesman Problem. As the derivations are similar to those presented herein, we refer the interested reader to [6] for further discussion.

Appendix

As noted in Subsection 3.3, inequalities (12) and (14) for the TSP generalize to the CVRP. These generalizations take into account the relative magnitudes of the demands q_i, q_j, and q_k. Consider Table 5 which is a modified, condensed version of Table 3 as follows. Column 1 of Table 5 refers to the same conditional logic statement numbers as column 1 of Table 3 though, for the reason explained below, only six such statements are present in Table 5. Columns 2 and 3 of Table 3 are not repeated in Table 5 since the antecedents and consequences are identical to those of Table 3 for each statement. Columns 2 through 5 of Table 5 give weakened consequences of column 3 of Table 3 in terms of the relative magnitudes of q_i, q_j, and q_k. Specifically, weakened consequences WC4(a), WC4(b), WC5(a), and WC5(b) of Table 5

Table 5: Additional Conditional Logic Statements for CVRP

No.	Weakened Consequence ($u_j - u_i \geq *$)			
	WC4(a) when $q_i \leq q_j$	WC4(b) when $q_i \geq q_j$	WC5(a) when $q_j \leq q_k$	WC5(b) when $q_j \geq q_k$
CL7	$\frac{q_i+q_j}{2}$	$\frac{-q_i+3q_j}{2}$	$\frac{3q_j-q_k}{2}$	$\frac{q_j+q_k}{2}$
CL8	$q_i + 2q_j + 2q_k - 2Q$	$3q_j + 2q_k - 2Q$	$2q_j - Q$	
CL9	$-q_j$			$-q_i - q_j + q_k$
CL10	$\frac{q_i+q_j}{2} - Q$	$\frac{-q_i+3q_j}{2} - Q$	$-q_i + \frac{5q_j-q_k}{2} - 2Q$	$-q_i + \frac{3q_j+q_k}{2} - 2Q$
CL11	$\frac{-q_i+q_j}{2} + q_k - 2Q$	$\frac{-3q_i+3q_j}{2} + q_k - 2Q$	$\frac{3q_j-q_k}{2} - Q$	$\frac{q_j+q_k}{2} - Q$
CL12		$-q_i + 2q_j + q_k - Q$	$q_i + 3q_j + q_k - 2Q$	$q_i + 2q_j + 2q_k - 2Q$
CL13	$q_i + q_j + q_k - 2Q$	$-q_i + 3q_j + q_k - 2Q$	$q_i + 3q_j - q_k - 2Q$	$q_i + q_j + q_k - 2Q$

require, respectively, that $q_i \leq q_j$, $q_i \geq q_j$, $q_j \leq q_k$, and $q_j \geq q_k$ as stated. For the rows of Table 5, we only list statements CL7 through CL13 since no other consequences are weakened from column 3 of Table 3.

To show the importance of the relative magnitudes of the demands in constructing Table 5, consider the consequence of statement CL7 of Table 3 stating that $u_j - u_i \geq q_j$ as found in column 3. The consequence of statement CL7 of WC4(a) found in column 2 of Table 5 stating that $u_j - u_i \geq \frac{q_i+q_j}{2}$ is valid when $q_j \geq \frac{q_i+q_j}{2}$ or, equivalently, when $q_i \leq q_j$.

Now consider the following, where consequences WC4(a) and WC4(b) yield generalizations of (12) and consequences WC5(a) and WC5(b) yield generalizations of (14).

(i) Suppose that $q_i \leq \min\{q_j, q_k\}$. Then we obtain the inequality

$$\begin{aligned} -2u_i + u_j + u_k \geq 2(q_i + q_j + q_k - 2Q) &+ \left(\frac{-q_i + q_j}{2} + 2Q\right) x_{ij} \\ &+ \left(\frac{-5q_i - 3q_k}{2} - 2q_j + 2Q\right) x_{ji} + \left(\frac{-q_i + q_k}{2} + 2Q\right) x_{ik} \\ &+ \left(\frac{-5q_i - 3q_j}{2} - 2q_k + 2Q\right) x_{ki} + \left(\frac{-3q_i - q_j}{2} - q_k + 2Q\right) x_{jk} \\ &+ \left(\frac{-3q_i - q_k}{2} - q_j + 2Q\right) x_{kj} \end{aligned} \tag{32}$$

using WC4(a) as follows. Since $q_i \leq q_j$, the consequences of WC4(a) are weaker than those of column 3 of Table 3, and they give the quadratic inequality

$$\begin{aligned} u_j - u_i \geq (q_i + q_j + q_k - 2Q) &+ \left(\frac{-q_i - q_j}{2} - q_k + 2Q\right) x_{ij} \\ &+ (-q_i - 2q_j - q_k + 2Q)x_{ji} + (q_j + q_k)x_{ik} + \left(\frac{-3q_i - q_j}{2}\right) x_{ki} \\ &+ \left(\frac{-q_i - q_j}{2} - q_k + Q\right) x_{jk} + (-q_i + Q)x_{kj} + T^3_{ijk}, \end{aligned} \tag{33}$$

where $T^3_{ijk} \equiv (Q - q_j - q_k)(x_{ik}x_{kj} - x_{ij}x_{jk}) + (-q_i - q_k)x_{ji}x_{ik} + (Q - q_k)x_{jk}x_{ki} + (q_i + q_j)x_{ki}x_{ij} + (q_j - Q)x_{kj}x_{ji}$. Since $q_i \leq q_k$, we can interchange the indices j and k within (33) to get a valid inequality in terms of $u_k - u_i$, and add this inequality to (33) to obtain (32), upon using that $T^3_{ijk} + T^3_{ikj} = 0$.

(ii) Suppose that $q_i \geq \max\{q_j, q_k\}$. Then we obtain the inequality

$$\begin{aligned} -2u_i + u_j + u_k \geq 2(-q_i + 2q_j + 2q_k - 2Q) + \left(\frac{3q_i - q_j}{2} - q_k + 2Q\right) x_{ij} \\ + \left(\frac{-q_i - 5q_k}{2} - 3q_j + 2Q\right) x_{ji} + \left(\frac{3q_i - q_k}{2} - q_j + 2Q\right) x_{ik} \\ + \left(\frac{-q_i - 5q_j}{2} - 3q_k + 2Q\right) x_{ki} + \left(\frac{q_i - 3q_j}{2} - 2q_k + 2Q\right) x_{jk} \\ + \left(\frac{q_i - 3q_k}{2} - 2q_j + 2Q\right) x_{kj} \end{aligned} \tag{34}$$

using WC4(b) as follows. Since $q_i \geq q_j$, the consequences of WC4(b) are weaker than those of column 3 of Table 3, and they give the quadratic inequality

$$\begin{aligned} u_j - u_i \geq (-q_i + 3q_j + q_k - 2Q) + \left(\frac{q_i - 3q_j}{2} - q_k + 2Q\right) x_{ij} \\ + (-3q_j - q_k + 2Q)x_{ji} + (q_i + q_k)x_{ik} + \left(\frac{-q_i - 3q_j}{2}\right) x_{ki} \\ + \left(\frac{q_i - 3q_j}{2} - q_k + Q\right) x_{jk} + (-q_j + Q)x_{kj} + T^3_{ijk}, \end{aligned} \tag{35}$$

where T^3_{ijk} is as defined above. Since $q_i \geq q_k$, we can interchange the indices j and k within (35) to get a valid inequality in terms of $u_k - u_i$, and add this inequality to (35) to obtain (34), again using that $T^3_{ijk} + T^3_{ikj} = 0$.

(iii) Suppose that $q_j \leq \min\{q_i, q_k\}$. Then we obtain the inequality

$$\begin{aligned} -u_i + 2u_j - u_k \geq 2(3q_j - 2Q) + \left(q_i + \frac{-3q_j + q_k}{2} + 2Q\right) x_{ij} \\ + \left(\frac{-3q_i - 7q_j}{2} - q_k + 2Q\right) x_{ji} + \left(\frac{-q_i - 5q_j}{2} + 2Q\right) x_{ik} \\ + \left(\frac{-5q_j - q_k}{2} + 2Q\right) x_{ki} + \left(-q_i + \frac{-7q_j - 3q_k}{2} + 2Q\right) x_{jk} \\ + \left(\frac{q_i - 3q_j}{2} + q_k + 2Q\right) x_{kj}. \end{aligned} \tag{36}$$

using WC5(a) as follows. Since $q_j \leq q_k$, the consequences of WC5(a) are weaker than those of column 3 of Table 3, and they give the quadratic inequality

$$\begin{aligned} u_j - u_i \geq (q_i + 3q_j - q_k - 2Q) + \left(-q_i + \frac{-3q_j + q_k}{2} + 2Q\right) x_{ij} \\ + (-2q_i - 3q_j + q_k + 2Q)x_{ji} + (-q_i - q_j + q_k + Q)x_{ik} \\ + \left(-q_i + \frac{-3q_j + q_k}{2} + Q\right) x_{ki} + \left(-2q_i + \frac{-q_j + q_k}{2}\right) x_{jk} \\ + (2q_k)x_{kj} + T^4_{ijk}, \end{aligned} \tag{37}$$

where $T^4_{ijk} \equiv (2q_i)x_{ij}x_{jk} + (-q_j - q_k + Q)x_{ik}x_{kj} + (q_i + q_j - q_k - Q)x_{ji}x_{ik} + (q_i - q_j - q_k + Q)x_{jk}x_{ki} + (q_i + q_j - Q)x_{ki}x_{ij} + (-2q_k)x_{kj}x_{ji}$. Since $q_j \leq q_i$, we can interchange the indices i and k within (37) to get a valid inequality in terms of $u_j - u_k$, and add this inequality to (37) to obtain (36), upon using that $T^4_{ijk} + T^4_{kji} = 0$.

(iv) Suppose that $q_j \geq \max\{q_i, q_k\}$. Then we obtain the inequality

$$\begin{aligned} -u_i + 2u_j - u_k \geq 2(q_i + q_j + q_k - 2Q) + \left(\frac{q_j - q_k}{2} + 2Q\right) x_{ij} \\ + \left(\frac{-5q_i - 3q_j}{2} - 2q_k + 2Q\right) x_{ji} + \left(\frac{-3q_i - q_j}{2} - q_k + 2Q\right) x_{ik} \\ + \left(-q_i + \frac{-q_j - 3q_k}{2} + 2Q\right) x_{ki} + \left(-2q_i + \frac{-3q_j - 5q_k}{2} + 2Q\right) x_{jk} \\ + \left(\frac{-q_i + q_j}{2} + 2Q\right) x_{kj}. \end{aligned} \tag{38}$$

using WC5(b) as follows. Since $q_j \geq q_k$, the consequences of WC5(b) are weaker than those of column 3 of Table 3, and they give the quadratic inequality

$$u_j - u_i \geq (q_i + q_j + q_k - 2Q) + \left(-q_i + \frac{-q_j - q_k}{2} + 2Q\right) x_{ij} + (-2q_i - 2q_j + 2Q)x_{ji} + (-q_i + Q)x_{ik} + \left(-q_i + \frac{-q_j - q_k}{2} + Q\right) x_{ki} + \left(-2q_i + \frac{q_j - q_k}{2}\right) x_{jk} + (q_j + q_k)x_{kj} + T^4_{ijk}, \quad (39)$$

where T^4_{ijk} is as defined above. Since $q_j \geq q_i$, we can interchange the indices i and k within (39) to get a valid inequality in terms of $u_j - u_k$, and add this inequality to (39) to obtain (38), again using that $T^4_{ijk} + T^4_{kji} = 0$.

For any (i, j, k) having $q_i = q_j = q_k$, inequalities (32) and (34) are the same, as are inequalities (36) and (38). Moreover, if $q_i = q_j = q_k = 1$ and $Q = n - 1$, then inequalities (32) and (34) each reduce to (12), and inequalities (36) and (38) each reduce to (14).

References

1. Bektaş, T., Gouveia, L.: Requiem for the Miller-Tucker-Zemlin subtour elimination constraints? European Journal of Operational Research **236**(3), 820–832 (2014). DOI 10.1016/j.ejor.2013.07.038
2. Campuzano, G., Obreque, C., Aguayo, M.M.: Accelerating the Miller-Tucker-Zemlin model for the asymmetric traveling salesman problem. Expert Systems with Applications **148**, 113229 (2020)
3. Christof, T., Löbel, A.: PORTA: POlyhedron Representation Transformation Algorithm. http://porta.zib.de (2004)
4. Dantzig, G., Fulkerson, D., Johnson, S.: Solution of a large-scale traveling-salesman problem. Journal of the Operations Research Society of America **2**(4), 393–410 (1954)
5. Desrochers, M., Laporte, G.: Improvements and extensions to the Miller-Tucker-Zemlin subtour elimination constraints. Operations Research Letters **10**(1), 27–36 (1991)
6. DeVries, A.: Tight representations of specially-structured 0-1 linear, quadratic, and polynomial programs. PhD dissertation, School of Mathematical and Statistical Sciences, Clemson University (2018)
7. Fischer, A.: An analysis of the asymmetric quadratic traveling salesman polytope. SIAM Journal on Discrete Mathematics **28**(1), 240–276 (2014)
8. Gouveia, L., Pires, J.M.: The asymmetric travelling salesman problem and a reformulation of the Miller-Tucker-Zemlin constraints. European Journal of Operational Research **112**(1), 134 – 146 (1999)
9. Gouveia, L., Pires, J.M.: The asymmetric travelling salesman problem: On generalizations of disaggregated Miller-Tucker-Zemlin constraints. Discrete Applied Mathematics **112**(1-3), 129 – 145 (2001)
10. Kara, I.: Tightening bounding constraints of the Miller-Tucker-Zemlin based formulation of the capacitated vehicle routing problems and some extensions. In: Proceeding of the 2nd International Conference on Manufacturing Engineering, Quality and Production Systems, pp. 137–142 (2010)
11. Kara, I., Laporte, G., Bektaş, T.: A note on the lifted Miller-Tucker-Zemlin subtour elimination constraints for the capacitated vehicle routing problem. European Journal of Operational Research **158**(3), 793–795 (2004)
12. Kulkarni, R., Bhave, P.: Integer progamming formulations of vehicle routing problems. European Journal of Operational Research **20**(1), 58–67 (1985)
13. Lougee-Heimer, R., Adams, W.: A conditional logic approach for strengthening mixed 0-1 linear programs. Annals of Operations Research **139**(1), 289–320 (2005)
14. Miller, C., Tucker, A., Zemlin, R.: Integer programming formulations and traveling salesman problems. Journal of the Association for Computing Machinery **7**(4), 326–329 (1960)
15. Sherali, H., Adams, W., Driscoll, P.: Exploiting special structures in constructing a hierarchy of relaxations for 0-1 mixed integer problems. Operations Research **46**(3), 396–405 (1998)
16. Sherali, H., Driscoll, P.: On tightening the relaxations of Miller-Tucker-Zemlin formulations for asymmetric traveling salesman problems. Operations Research **50**(4), 656–669 (2002)
17. Sherali, H.D., Adams, W.P.: A reformulation-linearization technique for solving discrete and continuous nonconvex problems, vol. 31. Springer Science & Business Media (2013)
18. Toth, P., Vigo, D.: Models, relaxations, and exact approaches for the capacitated vehicle routing problem. Discrete Applied Mathematics **123**(1-3), 487–512 (2002)