Convex Hull Results on Quadratic Programs with Non-Intersecting Constraints

Alex Joyce* Boshi Yang†

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Abstract

Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a set defined by quadratic constraints. Understanding the structure of the closed convex hull $\overline{\mathcal{C}}(\mathcal{F}) := \overline{\text{conv}}\,\{\,(x, xx^T) \mid x \in \mathcal{F}\,\}$ is crucial for solving quadratically constrained quadratic programs related to $\mathcal{F}$. An $\mathcal{F}$ with complicated structure can be constructed by intersecting simple sets. This paper discusses the relationship between $\overline{\mathcal{C}}(\mathcal{F})$ and $\overline{\mathcal{C}}(\mathcal{G})$, where $\mathcal{G}$ results by adding non-intersecting quadratic constraints to $\mathcal{F}$. We prove that $\overline{\mathcal{C}}(\mathcal{G})$ can be represented as the intersection of $\overline{\mathcal{C}}(\mathcal{F})$ and half spaces defined by the added constraints. The proof relies on a complete description of the asymptotic cones of sets defined by a single quadratic equality and a partial characterization of the recession cone of $\overline{\mathcal{C}}(\mathcal{F})$. Our proof generalizes an existing result for bounded $\mathcal{F}$ with non-intersecting hollows and several results on $\overline{\mathcal{C}}(\mathcal{F})$ for $\mathcal{F}$ defined by non-intersecting constraints. The result also implies a sufficient condition for when the closed convex hull of an intersection equals the intersection of the closed convex hulls.

Keywords: Convex hull, Non-intersecting, Semidefinite programming, Asymptotic cone, Quadratically constrained quadratic programming

1 Introduction

Let

$$\mathcal{F} := \{\, x \in \mathbb{R}^n \mid x^T A_i x + 2a_i^T x + \alpha_i \le 0,\ i \in I\,\}$$

be a quadratically defined set, where $I = \{1, \dots, m\}$, $A_i \in \mathcal{S}^n$, $a_i \in \mathbb{R}^n$, and $\alpha_i \in \mathbb{R}$ for $i \in I$. Here, $\mathcal{F}$ may be nonconvex and/or unbounded. In this paper, we are interested in

*School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, USA. Email: `joyce4@clemson.edu`.

†School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, USA. Email: `boshiy@clemson.edu`.

the structure of the closed convex hull

$$\overline{\mathcal{C}}(\mathcal{F}) := \overline{\text{conv}}\left\{ (x, xx^T) \;\middle|\; x \in \mathcal{F} \right\}.$$

$\overline{\mathcal{C}}(\mathcal{F})$ is closely related to quadratically constrained quadratic programming (QCQP). Specifically, let $Q \in \mathcal{S}^n$ and $c \in \mathbb{R}^n$, a QCQP problem minimizes a quadratic function of x over $\mathcal{F}$:

$$\begin{aligned} v := \min \quad & x^T Q x + 2c^T x \\ s.t. \quad & x \in \mathcal{F}. \end{aligned} \tag{1}$$

With a new matrix variable $X = xx^T$, the objective function of (1) can be linearized as $Q \bullet X + 2c^T x$, where $Q \bullet X := \sum_{i,j} Q_{ij} X_{ij}$ is the Frobenius product of Q and X. Due to the linearity of the objective function, one can convexify the feasible region and obtain the following well-known convex reformulation (e.g. [18, 13]):

$$\begin{aligned} v = \min \quad & Q \bullet X + 2c^T x \\ s.t. \quad & (x, X) \in \overline{\mathcal{C}}(\mathcal{F}). \end{aligned} \tag{2}$$

Despite being convex, (2) is computationally intractable due to the lack of explicit expression of $\overline{\mathcal{C}}(\mathcal{F})$. Therefore, understanding the structure of $\overline{\mathcal{C}}(\mathcal{F})$ is crucial for solving QCQP. Moreover, even a partial understanding of $\overline{\mathcal{C}}(\mathcal{F})$ is desired, as valid inequalities for $\overline{\mathcal{C}}(\mathcal{F})$ can help tighten the lower bounds of the convex relaxations of QCQP.

Since QCQP is NP-hard in general [19], it is unrealistic to expect a complete characterization of $\overline{\mathcal{C}}(\mathcal{F})$ for every quadratically defined set $\mathcal{F}$. On the other hand, $\overline{\mathcal{C}}(\mathcal{F})$ has been proved to be semidefinite representable in special cases. For instance, when $\mathcal{F}$ is defined by a single inequality [15, 21, 14, 22], a single equality [22, 24], or an interval-bounded inequality [20, 23], it is known that $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$, where

$$\mathcal{S}(\mathcal{F}) := \{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n \mid A_i \bullet X + 2a_i^T x + \alpha_i \le 0, \; i \in I, \; X \succeq xx^T \}$$

is the Shor relaxation of $\mathcal{F}$. Other semidefinite representable cases, for which $\overline{\mathcal{C}}(\mathcal{F}) \subsetneq \mathcal{S}(\mathcal{F})$, include when $\overline{\mathcal{C}}(\mathcal{F})$ is defined by a convex quadratic and multiple non-intersecting linear constraints [22, 26, 8, 17, 10], and when $\mathcal{F}$ is a low-dimensional polyhedron [1]. See [7] for a survey and the references therein. From a different but insightful prospective, the topic is also related to the discussion about when an semidefinite representable set is rank-1 generated [16, 2]. Even when $\overline{\mathcal{C}}(\mathcal{F})$ is not known to be semidefinite representable, the structure and applications of $\overline{\mathcal{C}}(\mathcal{F})$ are also widely studied. Related concepts in the literature include the cone of nonnegative quadratic functions [22], set-semidefiniteness [12, 13, 11], generalized copositivity [9], completely positivity over sets [4] and set-copositivity [6].

To better understand the structure of the closed convex hull of a set defined by complicated quadratic constraints, it is natural to decompose the constraints and consider the set as the intersection of two simpler sets. Specifically, let $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$, where

$$\mathcal{H} := \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \le 0,\ \ k \in K \,\}$$

is another quadratically defined set, $W_k \in \mathcal{S}^n$, $w_k \in \mathbb{R}^n$, $\omega_k \in \mathbb{R}$, and $K = \{1, \ldots, \ell\}$. We are interested in the relation between $\overline{\mathcal{C}}(\mathcal{G})$, $\overline{\mathcal{C}}(\mathcal{F})$ and $\overline{\mathcal{C}}(\mathcal{H})$. By definition, it is clear that

$$\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}).$$

On the other hand, $\overline{\mathcal{C}}(\mathcal{G})$ can be a proper subset of $\overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ in general. Such an example can be found even when $\mathcal{F}$ and $\mathcal{H}$ are as simple as two intersecting ellipsoids [8]. In this paper, we propose a sufficient condition for $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. In particular, we focus on the case when W_k's are nonzero and $\mathcal{H}$ induces non-intersecting constraints in $\mathcal{F}$. The nonzero requirement is technical (see Section 3) and the non-intersecting assumption is defined as follows.

Definition 1. *$\mathcal{H}$ is said to induce non-intersecting constraints in $\mathcal{F}$ if for all $k \in K$,*

$$x^T W_k x + 2w_k^T x + \omega_k = 0 \quad \Longrightarrow \quad \begin{cases} x^T A_i x + 2a_i^T x + \alpha_i \le 0, & \forall\ i \in I \\ x^T W_j x + 2w_j^T x + \omega_j \le 0, & \forall\ j \in K \setminus \{k\}. \end{cases} \tag{3}$$

For $k \in K$, let

$$\mathcal{H}_k := \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \le 0 \,\}.$$

Geometrically, $\mathcal{H} = \bigcap_{k \in K} \mathcal{H}_k$ induces non-intersecting constraints in $\mathcal{F}$ if and only if for each $k \in K$, the boundary of $\mathcal{H}_k$ is contained in $\mathcal{G}$.

Note that when $\mathcal{H}_k = \mathbb{R}^n$, the corresponding quadratic constraint is redundant, and when $\mathcal{H}_k = \emptyset$, the entire problem is not interesting. For the remaining nontrivial cases, the non-intersecting assumption (Definition 1) can be checked in polynomial time. Specifically, (3) holds if and only if

$$\max \{\, x^T A_i x + 2a_i^T x + \alpha_i \mid x^T W_k x + 2w_k^T x + \omega_k = 0 \,\} \le 0 \quad \forall\ i \in I, \tag{4}$$

$$\max \{\, x^T W_j x + 2w_j^T x + \omega_j \mid x^T W_k x + 2w_k^T x + \omega_k = 0 \,\} \le 0 \quad \forall\ j \in K \setminus \{k\}. \tag{5}$$

For each $k \in K$, let $h_k(x) := x^T W_k x + 2w_k^T x + \omega_k$. If there exist $\hat{x}$ and $\bar{x}$ such that $h_k(\hat{x}) < 0 < h_k(\bar{x})$, then the optimization problems in (4) and (5) enjoy exact Shor relaxations due to the S-Lemma with equality [24]. Therefore, the non-intersecting assumption (3) can be checked by solving semidefinite programs. On the other hand, if $h_k(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\mathcal{H}_k \neq \emptyset$, then $W_k \succeq 0$ and $w_k \in \text{Range}(W_k)$. In this case, the constraint $h_k(x) = 0$ is essentially an affine equality constraint. With suitable substitution, the optimization problems in (4) and (5) can be transformed to unconstrained quadratic problems and solved easily.

Concepts similar to Definition 1 have been mentioned in [25] and [2]. We restate those concepts here to avoid possible confusion. In [25], two linear constraints are called "non-intersecting" if the hyperplanes defined by the constraints do not intersect *inside the unit ball*. In [2], "non-interacting" constraints are explained as that if any of the constraints is active at a certain point x, then all the other constraints are satisfied *strictly* at x.

When $W_k \neq 0$ for all $k \in K$ and $\mathcal{H}$ induces non-intersecting constraints in $\mathcal{F}$, we show in this paper that $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ (Theorem 1), where

$$\mathcal{L}(\mathcal{H}) := \left\{ (x, X) \;\middle|\; W_k \bullet X + 2w_k^T x + \omega_k \leq 0,\; k \in K \right\},$$

and $\overline{\mathcal{C}}(\mathcal{H})$ is the same as the Shor relaxation $\mathcal{S}(\mathcal{H})$ when taking $\mathcal{F} = \mathbb{R}^n$ (Corollary 2). Since $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{S}(\mathcal{H})$, we conclude that $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ (Corollary 3). Our approach is motivated by the prior work on bounded quadratic programs with hollows [25]. When $\mathcal{F}$ is bounded, $\overline{\mathcal{C}}(\mathcal{F})$ is reduced to

$$\mathcal{C}(\mathcal{F}) := \text{conv} \left\{ (x, xx^T) \;\middle|\; x \in \mathcal{F} \right\}.$$

It is shown in [25] that $\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ if $\mathcal{F}$ is bounded and $\mathcal{H}$ induces non-intersecting constraints in $\mathcal{F}$. In this paper, we generalize the result to allow unbounded $\mathcal{F}$ and general nonzero W_k's. This generalization allows more intriguing applications. To illustrate, we provide two motivating examples.

Example 1. *Let* $\mathcal{F} := \{ x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0 \}$ *be the nonnegative quadrant and let* $\mathcal{H} := \{ x \in \mathbb{R}^2 \mid -(x_1 - x_2)^2 + 2x_2 - 1 \leq 0 \}$. $\mathcal{H}$ *induces a non-intersecting quadratic constraint in* $\mathcal{F}$ *as the boundary of* $\mathcal{H}$ *is contained in* $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$, *see Figure 1.*

Example 2. *Let* $\mathcal{G} := \{ x \in \mathbb{R}^2 \mid x_1 - x_2 \in \{-1, 0, 1\} \}$ *be a disjunctive set composed of the union of three parallel lines, see Figure 2. We can rewrite* $\mathcal{G}$ *as the intersection of* $\mathcal{F}$, $\mathcal{H}_1$

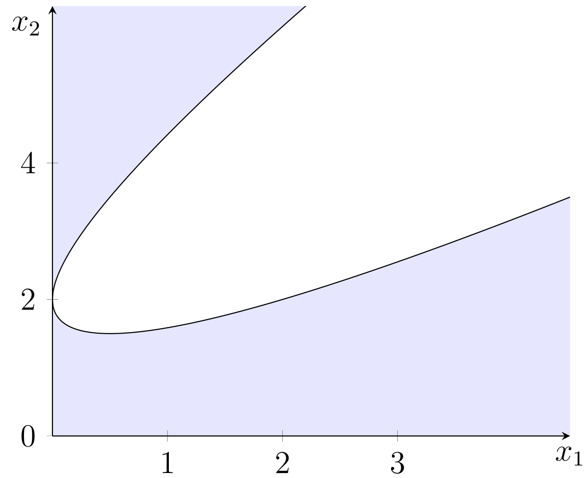

Figure 1: The first quadrant with a parabolic hollow.

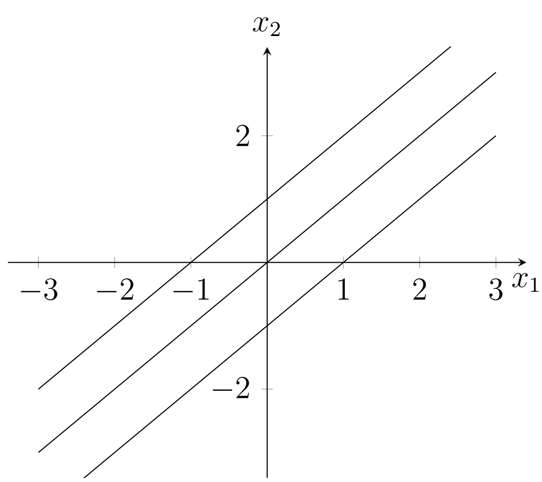

Figure 2: Three parallel lines.

and $\mathcal{H}_2$*, where*

$$\begin{aligned}\mathcal{F} &= \{\, x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 2x_1x_2 - 1 \le 0 \,\}\,,\\ \mathcal{H}_1 &= \{\, x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 2x_1x_2 - x_1 + x_2 \ge 0 \,\}\,,\\ \mathcal{H}_2 &= \{\, x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 2x_1x_2 + x_1 - x_2 \ge 0 \,\}\,.\end{aligned}$$

Here, $\mathcal{H} := \mathcal{H}_1 \cap \mathcal{H}_2$ *induces non-intersecting constraints in* $\mathcal{F}$*. Note that the closure of the complement of* $\mathcal{H}_i$ $(i = 1, 2)$ *is a maximal lattice-free set in* $\mathbb{R}^2$ *[5].*

We remark here that the proofs in [25] do not generalize to the unbounded case directly. In particular, two proofs are provided in [25]. The first one relies on discussing the locations of optimal solutions of $\min\{\, x^TQx + 2c^Tx \mid x \in \mathcal{G} \,\}$, while the second considers the extreme points of $\mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. For the first proof, when $\mathcal{F}$ is compact, an optimal solution of $\min \{ x^T Qx + 2c^T x \mid x \in \mathcal{G} \}$ always exists due to the Weierstrass extreme value theorem. However, in the unbounded case, an optimal solution may be unattainable even if the optimal value is finite. For the second, when $\mathcal{F}$ is bounded, $\mathcal{C}(\mathcal{F})$ is closed and is generated by its extreme points, which are in the form of (x, xx^T). When $\mathcal{F}$ is unbounded, $\mathcal{C}(\mathcal{F})$ is not necessarily closed, and $\overline{\mathcal{C}}(\mathcal{F})$ is generated by both its extreme points and its extreme directions. However, characterizing the extreme directions of $\overline{\mathcal{C}}(\mathcal{F})$ seems not to be an easy task.

To overcome the difficulty, we tailor a technical lemma by Dickinson et al. [11] to build a connection between $\overline{\mathcal{C}}(\mathcal{F})$ and $\mathcal{C}(\mathcal{F})$. As the connection is related to the asymptotic cone of $\mathcal{F}$, we introduce basic properties of the asymptotic cones and explore the structure of the asymptotic cones of sets defined by a quadratic equality in Section 2. In Section 3, we use the connection to build the proof of $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, for which two nontrivial pieces of the claim are considered sequentially. To show the necessity of the assumptions in Section 3, a counterexample is provided after the proof. Section 3 is concluded by two interesting corollaries, and the paper is concluded in Section 4.

Notation: For a nonempty set $S \subseteq \mathbb{R}^n$, we denote its boundary, interior, closure by $\operatorname{bd}(S)$, $\operatorname{int}(S)$, and $\overline{S}$, respectively. The (possibly nonconvex) cone generated by S is denoted by $\operatorname{cone}(S)$, and the conic hull of S is represented as $\operatorname{cone}\operatorname{conv}(S)$. Their closures are represented as $\overline{\operatorname{cone}}(S)$ and $\overline{\operatorname{cone}}\operatorname{conv}(S)$, respectively. The cardinality of S is denoted by $|S|$, and we use $\pm S$ to represent the set $S \cup (-S)$. When S is convex, the recession cone of S is denoted by $\operatorname{Rec}(S)$, and the set of extreme points of S is denoted by $\operatorname{ext}(S)$. For an m-by-n matrix P and a vector $v \in \mathbb{R}^n$, $PS := \{ Px \mid x \in S \}$ and $S \pm v := \{ x \pm v \mid x \in S \}$. The diagonal matrix whose diagonal entries are ordered components of a vector v is denoted by $\operatorname{Diag}(v)$.

2 Asymptotic cones

In this section, we focus on describing the unboundedness of a nonconvex set. We first introduce the asymptotic cones of nonempty sets.

For a nonempty set $S \subseteq \mathbb{R}^n$, the asymptotic cone of S is defined as

$$S_\infty := \left\{ d \in \mathbb{R}^n \;\middle|\; \exists\, t_k \to \infty, \{x^k\}_k \subseteq S \text{ such that } \lim_{k\to\infty} \frac{x^k}{t_k} = d \right\}.$$

S_∞ can be regarded as an extension of the recession cone of S. In particular, when S is convex, S_∞ coincides with the recession cone of S. We refer the readers to [3] for more detailed discussions about S_∞. In the following lemma, we list some properties of S_∞ which relate to our study.

Lemma 1 ([3]). *Let $S \subseteq \mathbb{R}^n$ be nonempty. Then,*

1. *S_∞ is a closed cone;*
2. *S is bounded if and only if $S_\infty = \{0\}$;*
3. *for any $\emptyset \neq T \subseteq S$, $T_\infty \subseteq S_\infty$;*
4. *for any $T \subseteq \mathbb{R}^n$ such that $S \cap T \neq \emptyset$, $(S \cap T)_\infty \subseteq S_\infty \cap T_\infty$;*
5. *if S is a closed convex set that contains no line, then $S = \mathrm{conv}(\mathrm{ext}(S)) + S_\infty$.*

The next lemma considers the asymptotic cone of S under invertible affine transformation. The proof of the lemma is straightforward based on the definition of S_∞ and is therefore omitted.

Lemma 2. *Let $V \in \mathbb{R}^{n\times n}$ be an invertible matrix, $v \in \mathbb{R}^n$ be a vector, and $S \subseteq \mathbb{R}^n$ be a nonempty set. Then,*

$$(VS)_\infty = V(S_\infty) \quad \text{and} \quad (S+v)_\infty = S_\infty.$$

2.1 Sets defined by one quadratic constraint

When S is a polyhedron, it is well known that S_∞ (i.e. $\mathrm{Rec}(S)$) can be described explicitly in a simple form. Unlike in the polyhedral case, we are not aware of any general characterization of S_∞ when S is defined by quadratic constraints. A special case is when S is defined by a single quadratic inequality. The following characterization of S_∞ is provided by Dickinson et al.

Proposition 1 ([11]). *For $S = \{\, x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \,\} \neq \emptyset$, we have*

$$S_\infty = \begin{cases} \{\, d \in \mathbb{R}^n \mid d^T Ad \leq 0 \,\}, & \text{if } A \not\succeq 0 \\ \{\, d \in \mathbb{R}^n \mid d^T Ad \leq 0,\ a^T d \leq 0 \,\} = \{\, d \in \mathbb{R}^n \mid Ad = 0,\ a^T d \leq 0 \,\}, & \text{if } A \succeq 0. \end{cases}$$

To facilitate the discussion in Section 3, we are interested in the form of S_∞ when S is defined by an *equality* constraint. Our result in the rest of the section can be regarded as a complement to Proposition 1 to fully characterize S_∞ when S is defined by a single quadratic (equality or inequality) constraint.

Consider $S = \{ x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha = 0 \}$. Let $A = V^T \Lambda V$ be an eigenvalue decomposition of A, where V is an orthogonal matrix and $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$. Since $S = \{ x \in \mathbb{R}^n \mid (Vx)^T \Lambda (Vx) + 2(Va)^T (Vx) + \alpha = 0 \}$,

$$VS = \left\{ y \in \mathbb{R}^n \;\middle|\; y^T \Lambda y + 2b^T y + \alpha = 0 \right\} = \left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \lambda_i y_i^2 + 2 \sum_{i=1}^n b_i y_i + \alpha = 0 \right\},$$

where $y = Vx$, $b = Va$, and their components are represented using subscripts. Moreover, let $Z := \{ i \mid \lambda_i = 0 \}$ be the index set of zero eigenvalues of A, then

$$VS = \left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i \left(y_i + \frac{b_i}{\lambda_i} \right)^2 + 2 \sum_{i \in Z} b_i y_i + \left(\alpha - \sum_{i \notin Z} \frac{b_i^2}{\lambda_i} \right) = 0 \right\}.$$

Define $\beta = \alpha - \sum_{i \notin Z} (b_i^2 / \lambda_i)$ and $v \in \mathbb{R}^n$ such that $v_i = 0$ for $i \in Z$ and $v_i = b_i$ for $i \notin Z$. Then,

$$VS + v = \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i z_i^2 + 2 \sum_{i \in Z} b_i z_i + \beta = 0 \right\}. \tag{6}$$

By Lemma 2, to characterize S_∞, we only need to focus on the case when A is diagonal and $a \in \mathrm{Null}(A)$. In addition to Z, we define

$$P := \{ i \mid \lambda_i > 0 \} \quad \text{and} \quad N := \{ i \mid \lambda_i < 0 \}$$

for easier reference. The following two lemmas consider the indefinite case and the semidefinite case, respectively.

Lemma 3. *For $S = \left\{ z \in \mathbb{R}^n \mid \sum_{i \notin Z} \lambda_i z_i^2 + 2 \sum_{i \in Z} b_i z_i + \beta = 0 \right\}$, if $P \neq \emptyset$ and $N \neq \emptyset$, then $S_\infty = \{ d \in \mathbb{R}^n \mid \sum_{i \notin Z} \lambda_i d_i^2 = 0 \}$.*

Proof. Let $D := \{ d \in \mathbb{R}^n \mid \sum_{i \notin Z} \lambda_i d_i^2 = 0 \}$. By Lemma 1 and Proposition 1,

$$\begin{aligned} S_\infty &\subseteq \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i z_i^2 + 2 \sum_{i \in Z} b_i z_i + \beta \leq 0 \right\}_\infty \bigcap \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i z_i^2 + 2 \sum_{i \in Z} b_i z_i + \beta \geq 0 \right\}_\infty \\ &= \left\{ d \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i d_i^2 \leq 0 \right\} \bigcap \left\{ d \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i d_i^2 \geq 0 \right\} \\ &= D. \end{aligned}$$

Now we consider the reverse direction. For any $d \in D$, if $d = 0$, the inclusion is trivially satisfied. If $d \neq 0$, we construct a sequence $\{z^k\}_k \subseteq S$ such that $\lim_{k \to \infty} \frac{z^k}{k} = d$. We consider two cases based on the emptiness of the index set $I := \{ i \in P \cup N \mid d_i \neq 0 \}$.

(i) If $I \neq \emptyset$, there exists $j \in P$ such that $d_j \neq 0$. (Otherwise, $\sum_{i \notin Z} \lambda_i d_i^2 < 0$.) Therefore, $\sum_{i \notin Z \cup \{j\}} \lambda_i d_i^2 = -\lambda_j d_j^2 < 0$. Let

$$\Delta^k := a_j^2 - \lambda_j \left(\left(\sum_{i \notin Z \cup \{j\}} \lambda_i d_i^2 \right) k^2 + 2 \left(\sum_{i \notin Z \cup \{j\}} b_i d_i \right) k + \beta \right).$$

Then there exists a large enough $\kappa > 0$ such that $\Delta^k > 0$ for all $k \geq \kappa$. For each $k \geq \kappa$, we define $z_i^k = k d_i$ for $i \neq j$ and $z_j^k = (-a_j + \sqrt{\Delta^k})/\lambda_j$.

(ii) If $I = \emptyset$, then $d \neq 0$ implies $Z \neq \emptyset$. If $\sum_{i \in Z} b_i d_i > 0$ or $\sum_{i \in Z} b_i d_i = 0$ and $\beta \geq 0$, there exists a $\kappa > 0$ such that $2k \sum_{i \in Z} b_i d_i + \beta \geq 0$ for all $k \geq \kappa$. In this case, we define z^k for $k \geq \kappa$ such that

$$z_i^k = \begin{cases} 0, & \text{if } i \in P \\ k d_i, & \text{if } i \in Z \\ \sqrt{-(2k \sum_{i \in Z} b_i d_i + \beta)/(|N| \lambda_i)}, & \text{if } i \in N. \end{cases}$$

If $\sum_{i \in Z} b_i d_i < 0$ or $\sum_{i \in Z} b_i d_i = 0$ and $\beta < 0$, there exists a $\kappa > 0$ such that $2k \sum_{i \in Z} b_i d_i + \beta \leq 0$ for all $k \geq \kappa$. In this case, for $k \geq \kappa$, we set

$$z_i^k = \begin{cases} \sqrt{-(2k \sum_{i \in Z} b_i d_i + \beta)/(|P| \lambda_j)}, & \text{if } i \in P \\ k d_i, & \text{if } i \in Z \\ 0, & \text{if } i \in N. \end{cases}$$

In each of the cases above, it is easy to check that $z^k \in S$ for $k \geq \kappa$ and $\lim_{k \to \infty} \frac{z^k}{k} = d$. Therefore, $D \subseteq S_\infty$. □

Lemma 4. *For $S = \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i \notin Z} \lambda_i z_i^2 + 2 \sum_{i \in Z} b_i z_i + \beta = 0 \right\} \neq \emptyset$, if $P \neq \emptyset$ and $N = \emptyset$, then $S_\infty = \{ d \in \mathbb{R}^n \mid d_i = 0$ for $i \in P$, $\sum_{i \in Z} b_i d_i \leq 0 \}$.*

Proof. If $Z = \emptyset$, then the statement holds trivially as S is bounded and $S_\infty = \{0\}$. Therefore, we assume $Z \neq \emptyset$. The proof is similar to that of Lemma 3. Let $D :=$

$\{ d \in \mathbb{R}^n \mid d_i = 0 \text{ for } i \in P,\ \sum_{i\in Z} b_i d_i \le 0 \}$. By Lemma 1 and Proposition 1,

$$\begin{aligned} S_\infty &\subseteq \left\{ z \in \mathbb{R}^n \;\middle|\; \sum_{i\notin Z} \lambda_i z_i^2 + 2\sum_{i\in Z} b_i z_i + \beta \le 0 \right\}_\infty \\ &= \left\{ d \in \mathbb{R}^n \;\middle|\; \sum_{i\notin Z} \lambda_i d_i^2 \le 0,\ \sum_{i\in Z} b_i d_i \le 0 \right\} \\ &= D. \end{aligned}$$

Here, the last equation holds because $N = \emptyset$.

Now we consider the reverse direction. For any $d \in D$, if $d = 0$, the inclusion is trivially satisfied. If $d \ne 0$, we construct a sequence $\{z^k\}_k \subseteq S$ such that $\lim_{k\to\infty} \frac{z^k}{k} = d$ as follows. If $\sum_{i\in Z} b_i d_i < 0$ or $\sum_{i\in Z} b_i d_i = 0$ and $\beta \le 0$, there exists a $\kappa > 0$ such that $2k\sum_{i\in Z} b_i d_i + \beta \le 0$ for all $k \ge \kappa$. In this case, for $k \ge \kappa$, we set

$$z_i^k = \begin{cases} \sqrt{-(2k\sum_{i\in Z} b_i d_i + \beta)/(|P|\lambda_j)}, & \text{if } i \in P \\ k d_i, & \text{if } i \in Z. \end{cases}$$

If $\sum_{i\in Z} b_i d_i = 0$ and $\beta > 0$, then $S \ne \emptyset$ implies that $b_j \ne 0$ for some $j \in Z$. In this case, we define z^k such that

$$z_i^k = \begin{cases} 0, & \text{if } i \in P \\ k d_i, & \text{if } i \in Z \setminus \{j\} \\ k d_j - \frac{\beta}{2b_j}, & \text{if } i = j. \end{cases}$$

One can check that $z^k \in S$ for $k \ge K$ and $\lim_{k\to\infty} \frac{z^k}{k} = d$. Therefore, $D \subseteq S_\infty$. □

Combining the results above, for $S = \{ x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha = 0 \}$, we provide an explicit description of S_∞ to end this section.

Proposition 2. *For $S = \{ x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha = 0 \} \ne \emptyset$, we have*

$$S_\infty = \begin{cases} \{ d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \le 0 \}, & \text{if } A \succeq 0 \text{ and } A \ne 0 \\ \{ d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \ge 0 \}, & \text{if } A \preceq 0 \text{ and } A \ne 0 \\ \{ d \in \mathbb{R}^n \mid d^T A d = 0 \}, & \text{if } A \text{ is indefinite} \\ \{ d \in \mathbb{R}^n \mid a^T d = 0 \}, & \text{if } A = 0. \end{cases}$$

Proof. When A is indefinite, by (6) and Lemma 3, we see that

$$(VS+v)_\infty = \{\, d \in \mathbb{R}^n \mid d^T \Lambda d = 0 \,\} = \{\, d \in \mathbb{R}^n \mid (V^T d)^T A (V^T d) = 0 \,\}.$$

By Lemma 2, $S_\infty = V^T(VS+v)_\infty = \{\, d \in \mathbb{R}^n \mid d^T A d = 0 \,\}$. Similarly, when $A \succeq 0$ and $A \neq 0$, by (6) and Lemma 4,

$$\begin{aligned}(VS+v)_\infty &= \{\, d \in \mathbb{R}^n \mid d^T \Lambda d = 0,\ b^T d \le 0 \,\} \\ &= \{\, d \in \mathbb{R}^n \mid (V^T d)^T A (V^T d) = 0,\ (V^T b)^T (V^T d) \le 0 \,\}.\end{aligned}$$

Therefore, Lemma 2 indicates that $S_\infty = V^T(VS+v)_\infty = \{\, d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \le 0 \,\}$. When $A \preceq 0$ and $A \neq 0$, since $S = \{\, x \in \mathbb{R}^n \mid x^T(-A)x - 2a^T x - \alpha = 0 \,\}$, the above result implies that

$$S_\infty = \{\, d \in \mathbb{R}^n \mid d^T(-A)d = 0,\ -a^T d \le 0 \,\} = \{\, d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \ge 0 \,\}.$$

Finally, when $A = 0$, S is polyhedral and $S_\infty = \operatorname{Rec}(S) = \{\, d \in \mathbb{R}^n \mid a^T d = 0 \,\}$. □

3 The closed convex hull result

In this section, we prove $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ under the non-intersecting assumption. We adopt the same notation mentioned in Section 1. Throughout this section, we make the following assumptions unless explicitly stated otherwise.

Assumption 1. *$\mathcal{H}$ induces non-intersecting constraints in $\mathcal{F}$. (See Definition 1.)*

Assumption 2. *$W_k \neq 0$ for all $k \in K$.*

We start from a simple observation. By definition, it is easy to check that

$$\operatorname{conv}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} = \operatorname{cone}\operatorname{conv}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} \cap \{\, Y \in \mathcal{S}^{n+1} \mid Y_{11} = 1 \,\},$$

where Y_{11} is the top left element of Y. We observe that the same statement also holds for the corresponding closures.

Lemma 5. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\operatorname{conv}}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} = \overline{\operatorname{cone}}\operatorname{conv}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} \cap \{\, Y \in \mathcal{S}^{n+1} \mid Y_{11} = 1 \,\}.$$

Proof. The forward containment "$\subseteq$" is straightforward. Now let Y be a matrix of the set on the right side of the equation. Then $Y_{11} = 1$, and there exists a sequence $\{Y_m\}_m$ such that $Y_m \to Y$ as $m \to \infty$ and

$$Y_m = \sum_{i=1}^{k_m} \lambda_{m_i} \begin{pmatrix} 1 \\ x_{m_i} \end{pmatrix} \begin{pmatrix} 1 \\ x_{m_i} \end{pmatrix}^T$$

for some $k_m \geq 0$, λ_{m_i} and $x_{m_i} \in S$. In particular, $\lambda_m := \sum_{i=1}^{k_m} \lambda_{m_i} \to 1$ as $m \to \infty$. Then $\tilde{Y}_m := Y_m/\lambda_m \in \operatorname{conv}\left\{ \binom{1}{x}\binom{1}{x}^T \mid x \in S \right\}$ and $\tilde{Y}_m \to Y$ as $m \to \infty$. Therefore, $Y \in \overline{\operatorname{conv}}\left\{ \binom{1}{x}\binom{1}{x}^T \mid x \in S \right\}$. ◻

The following lemma from [11] is crucial to characterize the closed convex hull.

Lemma 6 ([11])**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\operatorname{cone}} \operatorname{conv} \{ yy^T \mid y \in \{1\} \times S \} = \operatorname{conv} \{ yy^T \mid y \in \operatorname{cone}(\{1\} \times S) \cup (\{0\} \times S_\infty) \} .$$

Interpreting Lemma 6 by rewriting the equation in equivalent forms, we have the following lemma to characterize the difference between the convex hull $\mathcal{C}(S)$ and its closure $\overline{\mathcal{C}}(S)$.

Lemma 7. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\mathcal{C}}(S) = \mathcal{C}(S) + \operatorname{conv} \{ (0, dd^T) \mid d \in S_\infty \} .$$

Proof. By Lemma 6,

$$\begin{aligned} &\overline{\operatorname{cone}} \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} \\ &= \operatorname{conv} \left\{ yy^T \,\middle|\, y \in \operatorname{cone}(\{ 1 \} \times S) \cup (\{ 0 \} \times S_\infty) \right\} \\ &= \operatorname{conv} \left\{ yy^T \,\middle|\, y \in \operatorname{cone}(\{ 1 \} \times S) \right\} + \operatorname{conv} \left\{ yy^T \,\middle|\, y \in \{ 0 \} \times S_\infty \right\} \\ &= \operatorname{cone} \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} + \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^T \middle| \, d \in S_\infty \right\}. \end{aligned}$$

Intersecting both sides of the equation with $\{ Y \in \mathcal{S}^{n+1} \mid Y_{11} = 1 \}$, we have

$$\overline{\operatorname{conv}} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} = \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} + \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^T \middle| \, d \in S_\infty \right\}$$

by Lemma 5. Dropping the first component of the matrices, which is fixed to 1, the above equation can be equivalently written as $\overline{\mathcal{C}}(S) = \mathcal{C}(S) + \operatorname{conv} \{ (0, dd^T) \mid d \in S_\infty \}$. ◻

In fact, Lemma 7 builds a connection between $\mathrm{Rec}(\overline{\mathcal{C}}(S))$ and S_∞. Applying the description of the asymptotic cone in Section 2 to $\mathrm{bd}(\mathcal{H}_k) = \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k = 0 \,\}$, we have the following key observation.

Proposition 3. *If $d^T W_k d = 0$ for some $k \in K$, then $(0, dd^T) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$.*

Proof. Since $W_k \neq 0$ by Assumption 2, Proposition 2 indicates that $\{\, d \in \mathbb{R}^n \mid d^T W_k d = 0 \,\} = \pm(\mathrm{bd}(\mathcal{H}_k))_\infty$. By Assumption 1, $\mathrm{bd}(\mathcal{H}_k)$ is contained in $\mathcal{G}$. Consequently, $\pm(\mathrm{bd}(\mathcal{H}_k))_\infty \subseteq \pm\,\mathcal{G}_\infty$ by Lemma 1. Therefore, for any $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$, $\lambda \geq 0$ and $d \in \mathbb{R}^n$ such that $d^T W_k d = 0$,

$$\begin{aligned}(x, X) + \lambda(0, dd^T) &\in \overline{\mathcal{C}}(\mathcal{G}) + \mathrm{conv}\,\{\, (0, dd^T) \mid d \in \pm\,\mathcal{G}_\infty \,\} \\ &= \overline{\mathcal{C}}(\mathcal{G}) + \mathrm{conv}\,\{\, (0, dd^T) \mid d \in \mathcal{G}_\infty \,\} \\ &= \mathcal{C}(\mathcal{G}) + \mathrm{conv}\,\{\, (0, dd^T) \mid d \in \mathcal{G}_\infty \,\} = \overline{\mathcal{C}}(\mathcal{G}),\end{aligned}$$

where the second equation holds because of Lemma 7. That is, $(0, dd^T) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. □

As another technical lemma for the main proof, we restate the famous rank-1 decomposition by Sturm and Zhang.

Lemma 8 ([22])**.** *Let V be a symmetric matrix, and suppose $Y \succeq 0$ with $V \bullet Y = 0$ and $\mathrm{rank}(Y) = r$. Then there exists a rank-1 decomposition $Y = \sum_{i=1}^r y^i (y^i)^T$ such that $y^i \neq 0$ and $(y^i)^T V y^i = 0$ for all $i = 1, \ldots, r$.*

The proof of our main theorem is constructed by the following two propositions. In the first proposition, we show that $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathrm{bd}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$ by proving a more general statement.

Proposition 4. *If $X \succeq xx^T$ and $W_k \bullet X + 2w_k^T x + \omega_k = 0$ for some $k \in K$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. Since $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$ and $\begin{pmatrix} \omega_k & w_k^T \\ w_k & W_k \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 0$, by Lemma 8, there exist nonzero $y^j = \begin{pmatrix} z_0^j \\ z^j \end{pmatrix} \in \mathbb{R}^{1+n}$ for $j = 1, \ldots, r$ such that

$$0 = \begin{pmatrix} z_0^j \\ z^j \end{pmatrix}^T \begin{pmatrix} \omega_k & w_k^T \\ w_k & W_k \end{pmatrix} \begin{pmatrix} z_0^j \\ z^j \end{pmatrix} = (z^j)^T W_k z^j + 2 z_0^j w_k^T z^j + \omega_k (z_0^j)^2 \tag{7}$$

and

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{i=1}^r y^i (y^i)^T = \sum_{j \in J} (z_0^j)^2 \begin{pmatrix} 1 \\ x^j \end{pmatrix} \begin{pmatrix} 1 \\ x^j \end{pmatrix}^T + \sum_{j \notin J} \begin{pmatrix} 0 \\ z^j \end{pmatrix} \begin{pmatrix} 0 \\ z^j \end{pmatrix}^T,$$

where $J \coloneqq \{ j \mid y_1^j \neq 0 \}$ and $x^j = z^i/(z_0^j)$ for $j \in J$. Equivalently, $\sum_{j \in j}(z_0^j)^2 = 1$ and

$$(x, X) = \sum_{j \in J}(z_0^j)^2(x^j, x^j(x^j)^T) + \sum_{j \notin J}(0, z^j(z^j)^T).$$

By (7), $(z^j)^T W_k z^j = 0$ for $j \notin J$. Therefore, Proposition 3 indicates that $\sum_{j \notin J}(0, z^j(z^j)^T) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. Also by (7), $x^j \in \mathrm{bd}(\mathcal{H}_k) \subseteq \mathcal{G}$ for all $j \in J$. Therefore, $(x, X) \in \mathcal{C}(\mathcal{G}) + \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. □

We remark here that when W_k is (positive or negative) definite, $|J| = r$ and the proof of Proposition 4 reduces to the alternative proof of Corollary 1 in [25]. When W_k is not definite, the term $\sum_{j \notin j}(0, z^j(z^j)^T)$ is related to $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$ in our proof by the prior discussion on the asymptotic cones, which helps generalizing the result.

Leveraging Proposition 4, we show in the following proposition that $\mathcal{C}(\mathcal{F}) \cap \mathrm{int}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. In [25], this case is trivial as it suffices to consider the extreme points of $\mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ when $\mathcal{F}$ is compact. Here, we need to adopt a different approach due to the unboundedness of $\mathcal{F}$. We consider a generic point (x, X) in $\mathcal{C}(\mathcal{F}) \cap \mathrm{int}(\mathcal{L}(\mathcal{H}))$ and decompose it into "rank-1" points in $\mathcal{C}(\mathcal{F})$. If all the "rank-1" points are in $\mathcal{L}(\mathcal{H})$, then (x, X) is in $\mathcal{C}(\mathcal{G})$; if some "rank-1" point is not in $\mathcal{L}(\mathcal{H})$, we take a convex combination of the point and (x, X) to "drag" it back to $\mathcal{L}(\mathcal{H})$.

Proposition 5. *If $(x, X) \in \mathcal{C}(\mathcal{F})$ and $W_k \bullet X + 2w_k^T x + \omega_k < 0$ for all $k \in K$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. Since $(x, X) \in \mathcal{C}(\mathcal{F})$, there exist $x^j \in \mathcal{F}$, $\mu_j > 0$ for $j = 1, \ldots, p$, such that $\sum \mu_j = 1$ and

$$(x, X) = \sum_{j=1}^{p} \mu_j(x^j, x^j(x^j)^T).$$

Let $J_k \coloneqq \{ j \mid (x^j)^T W_k x^j + 2w_k^T x^j + \omega_k \leq 0 \}$. If $|J_k| = p$ for all $k \in K$, then $x^j \in \mathcal{F} \cap \mathcal{H} = \mathcal{G}$ for $j = 1, \ldots, p$, and therefore $(x, X) \in \mathcal{C}(\mathcal{G}) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. If $|J_k| < p$ for some $k \in K$, then for each $j \notin J_k$, we have $W_k \bullet (x^j(x^j)^T) + 2w_k^T x^j + \omega_k > 0$. Since $\{ (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_k = 0 \}$ separates $(x^j, x^j(x^j)^T)$ and (x, X), there exists $\gamma_j \in (0, 1)$ such that

$$(\hat{x}^j, \hat{X}^j) \coloneqq \gamma_j(x, X) + (1 - \gamma_j)(x^j, x^j(x^j)^T) \in \mathcal{C}(\mathcal{F})$$

satisfies $W_k \bullet \hat{X}^j + w_k^T \hat{x}^j + \omega_k = 0$. Since $\hat{X}^j \succeq \hat{x}^j(\hat{x}^j)^T$, Proposition 4 indicates that

$(\hat{x}^j, \hat{X}^j) \in \overline{\mathcal{C}}(\mathcal{G})$. By the definition of $(\hat{x}^j, \hat{X}^j)$,

$$
\begin{aligned}
(x, X) &= \sum_{j \in J_k} \mu_j (x^j, x^j (x^j)^T) + \sum_{j \notin J_k} \mu_j (x^j, x^j (x^j)^T) \\
&= \sum_{j \in J_k} \mu_j (x^j, x^j (x^j)^T) + \sum_{j \notin J_k} \mu_j \left(\frac{1}{1-\gamma_j} (\hat{x}^j, \hat{X}^j) - \frac{\gamma_j}{1-\gamma_j} (x, X) \right).
\end{aligned}
$$

Let $\sigma := 1 + \sum_{j \notin J_k} \frac{\mu_j \gamma_j}{1-\gamma_j}$, then

$$
(x, X) = \sum_{j \in J_k} \frac{\mu_j}{\sigma} (x^j, x^j (x^j)^T) + \sum_{j \notin J_k} \frac{\mu_j}{\sigma (1-\gamma_j)} (\hat{x}^j, \hat{X}^j),
$$

which is a convex combination of points in $\overline{\mathcal{C}}(\mathcal{G})$. Therefore, $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$. $\square$

Using a continuity argument, Proposition 5 can be generalized to $\overline{\mathcal{C}}(\mathcal{F}) \cap \operatorname{int}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$.

Corollary 1. *If $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$ and $W_k \bullet X + 2 w_k^T x + \omega_k < 0$ for all $k \in K$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. If $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$, there exists a sequence $\{ (x^t, X^t) \}_t \subseteq \mathcal{C}(\mathcal{F})$ such that $(x^t, X^t) \to (x, X)$ as $t \to \infty$. Since $W_k \bullet X + 2 w_k^T x + \omega_k < 0$, for sufficiently large t, $W_k \bullet X^t + 2 w_k^T x^t + \omega_k < 0$. By Proposition 5, $(x^t, X^t) \in \overline{\mathcal{C}}(\mathcal{G})$. The proof is completed by taking $t \to \infty$. $\square$

Summarizing the above, we state the main theorem of this section as follows.

Theorem 1. *Under Assumptions 2 and 1, $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$.*

Proof. The forward direction "$\subseteq$" is easy since $\overline{\mathcal{C}}(\mathcal{G}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. The other direction is given by combining Proposition 4 and Corollary 1. $\square$

The non-intersecting assumption (Assumption 1) is essential in Theorem 1. We refer the readers to [25] for counterexamples when $W_k \succ 0$ and the non-intersecting assumption is missing. The following example shows that the nonzero requirement (Assumption 2) cannot be dropped.

Example 3. *Let $\mathcal{F} = \{ x \in \mathbb{R} \mid -x - 2 \leq 0 \} = [-2, \infty)$ and $\mathcal{H} = \{ x \in \mathbb{R} \mid -x + 1 \geq 0 \} = (-\infty, 1]$. Obviously, $\mathcal{H}$ induces a non-intersecting constraint in $\mathcal{F}$. Note that for $\mathcal{G} := \mathcal{F} \cap \mathcal{H} = [-2, 1]$,*

$$
\overline{\mathcal{C}}(\mathcal{G}) = \{ (x, X) \in \mathbb{R}^2 \mid X \leq 2 - x, X \geq x^2 \},
$$

which is a proper subset of

$$
\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \{ (x, X) \in \mathbb{R}^2 \mid -2 \leq x \leq 1, X \geq x^2 \}.
$$

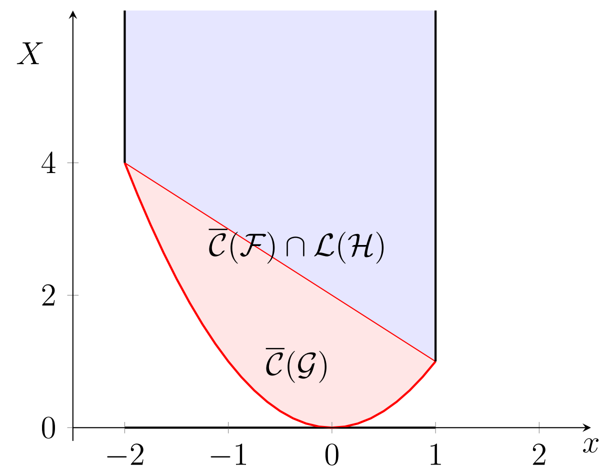

Figure 3: In Example 3, $\overline{\mathcal{C}}(\mathcal{G})$ is a bounded set while $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ is unbounded.

We conclude this section with two corollaries of Theorem 1. The first corollary shows that $\overline{\mathcal{C}}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$ when $\mathcal{H}$ is defined by non-intersecting quadratic constraints. Special cases and variants of the corollary can be spotted in the literature. To name a few: the non-binding constraints in [26], the generalized trust region subproblem in [20], and the non-interacting constraints in [2].

Corollary 2. *Let $\mathcal{H} = \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \le 0,\ k \in K \,\}$ be set defined by non-intersecting quadratic inequalities. That is, $W_k \ne 0$ for all $k \in K$ and for all $x \in \mathbb{R}^n$,*

$$x^T W_k x + 2w_k^T x + \omega_k = 0 \quad \Longrightarrow \quad x^T W_j x + 2w_j^T x + \omega_j \le 0 \ \ \forall\, j \in K \setminus \{k\}.$$

Then, $\overline{\mathcal{C}}(\mathcal{H}) = \mathcal{S}(\mathcal{H}) = \{\, (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_i \le 0,\, k \in K,\ X \succeq xx^T \,\}$.

Proof. Note that $\mathcal{H}$ satisfies Assumptions 1 and 2 with $\mathcal{F} = \mathbb{R}^n$. Therefore,

$$\overline{\mathcal{C}}(\mathcal{H}) = \overline{\mathcal{C}}(\mathbb{R}^n) \cap \mathcal{L}(\mathcal{H}) = \{\, (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_i \le 0,\, k \in K,\ X \succeq xx^T \,\} = \mathcal{S}(\mathcal{H}).$$

□

The second corollary can be interpreted as a sufficient condition for $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. Note that $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{S}(\mathcal{H})$. By Theorem 1 and Corollary 2, we have the following statement.

Corollary 3. *Under Assumptions 1 and 2, $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$.*

4 Conclusion

For quadratically defined sets $\mathcal{F}$ and $\mathcal{H}$, we consider the relation between $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H})$ and $\overline{\mathcal{C}}(\mathcal{F})$. We show that $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ when $\mathcal{H}$ induces non-intersecting constraints in $\mathcal{F}$ and the quadratic functions defining $\mathcal{H}$ have nonzero Hessians. This result generalizes the bounded case in [25] and other non-intersecting cases captured by Corollary 2. To prove the result, a complete characterization of the asymptotic cones of sets defined by a single quadratic equality is provided as a byproduct.

The convex hull result can be uniformly applied to any quadratically defined set $\mathcal{F}$ with known closed convex hull $\overline{\mathcal{C}}(\mathcal{F})$ to generate new convex hull results. The result can also be interpreted as a sufficient condition for $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. For future research, it is worthwhile to explore other sufficient conditions and necessary conditions for $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. It would also be interesting to see how the convex hull result can help in the computation of mixed-integer QCQP.

To conclude this paper, we bring to the reader's attention a simple but interesting example, which is beyond the reach of the results in this paper. Let $\mathcal{G} := \{\, x \in \mathbb{R}^2 \mid x_1x_2 \le 1, x \ge 0 \,\}$. Note that $\mathcal{H} := \{\, x \in \mathbb{R}^2 \mid x_1x_2 \le 1 \,\}$ has two branches, and the boundary of the branch in the 3rd quadrant is not contained in $\mathcal{G}$. Therefore, $\mathcal{H}$ does not induce non-intersecting constraints in $\mathcal{F} := \{\, x \in \mathbb{R}^n \mid x \ge 0 \,\}$. Such cases are worth exploring as bilinear terms are fundamental substructures in mixed-integer QCQP. Future research along the path is desired for cases when the non-intersecting assumption is relaxed.

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