On the time-dependent behavior of homogeneous Quasi-Birth-Death processes with finitely many levels

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Abstract

It is known at this point that the Laplace transforms of the transition functions of a Quasi-Birth-Death (QBD) process with a single boundary level exhibits a matrix-geometric structure, and can be expressed in terms of a type of $\mathbf{R}$-matrix and $\mathbf{G}$-matrix that are analogous to the classical $\mathbf{R}$ and $\mathbf{G}$ matrices found in the stationary distribution of this process. Our objective is to provide a new study of the time-dependent behavior of similar QBD processes that have two boundary levels. Through completely probabilistic methods (i) we study the distribution of the amount of time it takes such a QBD process to move from one level to another level, and (ii) we show how the Laplace transforms of the transition functions of such a QBD process can be expressed entirely in terms of simpler $\mathbf{R}$-matrices that appear in the Laplace transforms of the transition functions of two different, but related QBD processes having infinitely many levels.

Keywords: matrix-analytic methods; matrix-geometric methods; quasi-birth-death processes; time-dependent behavior.
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1 Introduction

In 1982, Hajek [9] showed that the stationary distribution of a homogeneous Quasi-Birth-Death (QBD) process having finitely many levels exhibits its own type of 'matrix-geometric' form that contains two different types of $\mathbf{R}$-matrices associated with those that appear in the stationary distribution of homogeneous QBD processes having infinitely many levels and a single boundary level. Fifteen years later, Keilson and Masuda showed in [13] that the Laplace transforms of the transition functions of a homogeneous QBD process also exhibit an analogous type of 'matrix-geometric' form, but to get this form the authors make use of what they refer to as a 'compensation method' which is very analytic in flavor, and appears to be quite different from the approach used by Hajek in [9]: this compensation method was also used in Keilson and Zachmann [14] to study stationary distributions associated with these processes. The approach given in [13] also appears to be somewhat incomplete, as they leave open the problem of calculating certain Laplace transforms associated with the boundary levels of the QBD process.

Our objective is to provide a completely probabilistic approach towards deriving such matrix-geometric expressions for the transition functions of a homogeneous QBD process having finitely many levels, while simultaneously showing how to numerically calculate all involved Laplace transforms, including those associated with the boundary levels. It is important to point out that the formulas we derive for the Laplace transforms of the transition functions associated with a homogeneous, finite QBD process are similar in flavor to quantities recently derived in Dendievel et al [4], where the authors were instead interested in studying the behavior of a reward function associated with a homogeneous QBD process having finitely many levels, but in the analysis found in [4] the authors rely extensively on the theory of matrix difference equations, whereas our approach avoids usage of this theory entirely.

The key to deriving our main results involves first deriving, through entirely probabilistic methods, the joint distribution of the amount of time it takes a homogeneous QBD process to reach, from a given level n, either a level $a < n$ or a level $b > n$, as well as the level and phase of the process at this random time: these distributions can be fully described with two types of '**G**-matrices' associated with the QBD process. This distribution can be studied probabilistically with the strong Markov property, through a matrix generalization of an argument found in Doroudi et al [5] in the context of M/M/1 queues. Interestingly, the same type of proof technique can be used to derive simple expressions for two different types of **R**-matrices that, given levels a, n, b satisfying $a < n < b$, keep track of the (discounted) expected amount of time spent by the QBD process in state (n, j) for some phase j before the chain revisits either level a or level b, given it starts either at some state in level a, or some state in level b. It may seem strange at first glance that the same proof technique can be used to derive these type of **G** matrices and these type of **R** matrices, but what makes this possible is the fact that each element of these **R**-matrices can be expressed in terms of the expected value of a 'random-product' governed by an alternative CTMC related to the original QBD process. Readers wishing to read more about the random-product technique itself should consult [3, 6, 7]: moreover, [11] also shows how the random-product technique can be used to derive the Laplace transforms of the transition functions of a QBD process with a single boundary.

It is also important to observe that our overall approach appears to yield new results that address the amount of time it takes a homogeneous QBD process with finitely many levels to move from one level to another. Properties of these hitting-time distributions have been studied by numerous authors in both the discrete-time and continuous-time context, see e.g. [8], [15], and [17], but to the best of our knowledge our approach appears to yield new expressions for the Laplace-Stieltjes transforms of these hitting-time random variables.

2 Two Important Lemmas

Here we state and prove two useful lemmas that provide us with computable expressions for the matrices needed in order to derive our main results.

Suppose $\{F(t); t \geq 0\}$ is an irreducible Quasi-Birth-Death (QBD) process, having a state space S of the form

$$S = \bigcup_{n \in \mathbb{Z}} L_n$$

where for each integer $n \in \mathbb{Z}$, $L_n = \{(n, 1), (n, 2), \ldots, (n, d)\}$ for some fixed integer $d \geq 1$. The transition rate matrix $\mathbf{Q}$ of $\{F(t); t \geq 0\}$ also exhibits a block-partitioned structure that is constructed using only three matrices $\mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1 \in \mathbb{R}^{d \times d}$, where for each $i, j \in \{1, 2, \ldots, d\}$ (where possibly $i = j$), and each $n \in \mathbb{Z}$,

$$q((n,i),(n-1,j)) = (\mathbf{A}_{-1})_{i,j}, \quad q((n,i),(n,j)) = (\mathbf{A}_0)_{i,j}, \quad q((n,i),(n+1,j)) = (\mathbf{A}_1)_{i,j}$$

and for any two integers n, m satisfying $|n - m| \geq 2$, $q((n,i),(m,j)) = 0$. for each $i, j \in \{1, 2, \ldots, d\}$.

We further associate with $\{F(t); t \geq 0\}$ hitting-time random variables of the form τ_A, where for each $A \subset S$,

$$\tau_A := \inf\{t \geq 0 : F(t) \in A\}.$$

From these hitting times, we construct the matrices $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$, where for each $i, j \in \{1, 2, \ldots, d\}$, the (i, j)th element found in the matrix $\mathbf{G}(\alpha)$ is

$$(\mathbf{G}(\alpha))_{i,j} := \mathbb{E}_{(0,i)}[e^{-\alpha \tau_{L_{-1}}} \mathbf{1}(F(\tau_{L_{-1}}) = (-1, j))], \quad (\hat{\mathbf{G}}(\alpha))_{i,j} := \mathbb{E}_{(0,i)}[e^{-\alpha \tau_{L_1}} \mathbf{1}(F(\tau_{L_1}) = (1, j))].$$

The level-independent structure of $\mathbf{Q}$ reveals that for each integer $n \geq 1$,

$$(\mathbf{G}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_{n-1}}}\mathbf{1}(F(\tau_{L_{n-1}}) = (n-1,j))], \quad (\hat{\mathbf{G}}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_{n+1}}}\mathbf{1}(F(\tau_{L_{n+1}}) = (n+1,j))].$$

Furthermore, we can use the Strong Markov property to show that for each $a, b \in \mathbb{Z}$ satisfying $a < b$,

$$(\mathbf{G}(\alpha)^{b-a})_{i,j} = \mathbb{E}_{(b,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(F(\tau_{L_a}) = (a,j))], \quad (\hat{\mathbf{G}}(\alpha)^{b-a})_{i,j} = \mathbb{E}_{(a,i)}[e^{-\alpha\tau_{L_b}}\mathbf{1}(F(\tau_{L_b}) = (b,j))].$$

Fix two integers $a, b \in \mathbb{Z}$, where $a < b$. Together the matrices $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$ can be used to construct the matrices $\mathbf{G}_{n,a,b}(\alpha)$ and $\hat{\mathbf{G}}_{n,b,a}(\alpha)$, where for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{G}_{n,a,b}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(\tau_{L_a} < \tau_{L_b}, F(\tau_{L_a}) = (a,j))]$$

and

$$(\hat{\mathbf{G}}_{n,b,a}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_b}}\mathbf{1}(\tau_{L_b} < \tau_{L_a}, F(\tau_{L_b}) = (b,j)].$$

The next lemma, Lemma 2.1, shows that both of these matrices can be expressed explicitly in terms of $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$. This lemma is very similar to an exercise found in Karlin and Taylor [12] pertaining to Brownian motion, and it is also similar to a result in the work of Doroudi et al [5], which addresses analogous hitting-time results associated with a process that is the difference of two independent, homogeneous Poisson processes.

Lemma 2.1 *Given $a, n, b \in \mathbb{Z}$ satisfying $a < n < b$, we have*

$$\mathbf{G}_{n,a,b}(\alpha) = [\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-n}\mathbf{G}(\alpha)^{b-n}]\mathbf{G}(\alpha)^{n-a}[\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-a}\mathbf{G}(\alpha)^{b-a}]^{-1}. \tag{1}$$

Moreover,

$$\hat{\mathbf{G}}_{n,b,a}(\alpha) = [\mathbf{I} - \mathbf{G}(\alpha)^{n-a}\hat{\mathbf{G}}(\alpha)^{n-a}]\hat{\mathbf{G}}(\alpha)^{b-n}[\mathbf{I} - \mathbf{G}(\alpha)^{b-a}\hat{\mathbf{G}}(\alpha)^{b-a}]^{-1}. \tag{2}$$

Proof Fix $i, j \in \{1, 2, \ldots, d\}$, and observe first that

$$\begin{aligned}
&(\mathbf{G}(\alpha)^{n-a})_{i,j} \\
&= \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(F(\tau_{L_a}) = (a,j))] \\
&= \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(F(\tau_{L_a}) = (a,j), \tau_{L_a} < \tau_{L_b})] + \sum_{\nu=1}^{m}\mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_a}}\mathbf{1}(F(\tau_{L_a}) = (a,j), F(\tau_{L_b}) = (b,\nu), \tau_{L_b} < \tau_{L_a})] \\
&= (\mathbf{G}_{n,a,b}(\alpha))_{i,j} + \sum_{\nu=1}^{m}(\hat{\mathbf{G}}_{n,b,a}(\alpha))_{i,\nu}(\mathbf{G}(\alpha)^{b-a})_{\nu,j}
\end{aligned}$$

which, in matrix form, is simply

$$\mathbf{G}(\alpha)^{n-a} = \mathbf{G}_{n,a,b}(\alpha) + \hat{\mathbf{G}}_{n,b,a}(\alpha)\mathbf{G}(\alpha)^{b-a}.$$

A similar argument further reveals that

$$\hat{\mathbf{G}}(\alpha)^{b-n} = \mathbf{G}_{n,a,b}(\alpha)\hat{\mathbf{G}}(\alpha)^{b-a} + \hat{\mathbf{G}}_{n,b,a}(\alpha).$$

Solving this resulting system consisting of two matrix equations with two matrix unknowns, while making use the fact that $(\mathbf{I} - \mathbf{G}(\alpha)^{b-a}\hat{\mathbf{G}}(\alpha)^{b-a})^{-1}$ and $(\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-a}\mathbf{G}(\alpha)^{b-a})^{-1}$ exist due to both $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$ having spectral radius strictly less than one, yields

$$\mathbf{G}_{n,a,b}(\alpha) = [\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-n}\mathbf{G}(\alpha)^{b-n}]\mathbf{G}(\alpha)^{n-a}[\mathbf{I} - \hat{\mathbf{G}}(\alpha)^{b-a}\mathbf{G}(\alpha)^{b-a}]^{-1}$$

and

$$\hat{\mathbf{G}}_{n,b,a}(\alpha) = [\mathbf{I} - \mathbf{G}(\alpha)^{n-a}\hat{\mathbf{G}}(\alpha)^{n-a}]\hat{\mathbf{G}}(\alpha)^{b-n}[\mathbf{I} - \mathbf{G}(\alpha)^{b-a}\hat{\mathbf{G}}(\alpha)^{b-a}]^{-1}$$

which proves the claim. □

A similar type of result also holds within the context of $\mathbf{R}$-matrices. For each subset $A \subset \mathbb{Z}$, each integer $m \in A$, and each integer $n \in A^c$, we define the matrix $\mathbf{R}_{m,A,n}(\alpha)$ as follows: for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{R}_{m,A,n}(\alpha))_{i,j} := (q((m,i)) + \alpha)\mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_A}} e^{-\alpha t}\mathbf{1}(F(t) = (n,j))dt\right]$$

where $L_A := \bigcup_{m\in A} L_m$.

Given the homogeneous structure present among the block structure of $\mathbf{Q}$, it is well-known (see e.g. [11]) that for each $m \in \mathbb{Z}$, and each $n > m$, that

$$\mathbf{R}_{m,\{m\},n}(\alpha) = \mathbf{R}_{0,\{0\},1}(\alpha)^{n-m}.$$

Likewise, for each $m \in \mathbb{Z}$ and each $n < m$, we have

$$\mathbf{R}_{m,\{m\},n}(\alpha) = \mathbf{R}_{0,\{0\},-1}(\alpha)^{m-n}$$

so it is useful to define the matrices $\mathbf{R}(\alpha)$ and $\hat{\mathbf{R}}(\alpha)$ as

$$\mathbf{R}(\alpha) := \mathbf{R}_{0,\{0\},1}(\alpha), \quad \hat{\mathbf{R}}(\alpha) := \mathbf{R}_{0,\{0\},-1}(\alpha).$$

Our next result, Lemma 2.2, provides us with a way of expressing, for $a < n < b$, the matrices $\mathbf{R}_{a,\{a,b\},n}(\alpha)$ and $\mathbf{R}_{b,\{a,b\},n}(\alpha)$ in terms of $\mathbf{R}(\alpha)$ and $\hat{\mathbf{R}}(\alpha)$. Readers should compare the proof we provide of this result with the proof of Lemma 10.3.1 from [16], which instead addresses the case where $\alpha = 0$ (and instead addresses the discrete-time case).

Lemma 2.2 *Fix two integers a, b such that $a < b$. Then for each integer $n \in \{a+1, a+2, \ldots, b-2, b-1\}$, we have*

$$\mathbf{R}_{a,\{a,b\},n}(\alpha) = (\mathbf{I} - \mathbf{R}(\alpha)^{b-a}\hat{\mathbf{R}}(\alpha)^{b-a})^{-1}\mathbf{R}(\alpha)^{n-a} - (\mathbf{I} - \mathbf{R}(\alpha)^{b-a}\hat{\mathbf{R}}(\alpha)^{b-a})^{-1}\mathbf{R}(\alpha)^{b-a}\hat{\mathbf{R}}(\alpha)^{b-n} \tag{3}$$

and

$$\mathbf{R}_{b,\{a,b\},n}(\alpha) = -(\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}(\alpha)^{b-a})^{-1}\hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}(\alpha)^{n-a} + (\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{b-a}\mathbf{R}(\alpha)^{b-a})^{-1}\hat{\mathbf{R}}(\alpha)^{b-n}. \tag{4}$$

Proof This result can be established though usage of the random-product technique: see [3, 6, 7], as well as [11] for an example of how the random-product technique was first applied to the theory of Markov processes of G/M/1-type. In order to use the random-product technique, we associate with $\{F(t); t \geq 0\}$ an alternative CTMC $\{\tilde{F}(t); t \geq 0\}$ whose generator $\tilde{\mathbf{Q}}$ satisfies two properties:

(i) For each $x, y \in S$ satisfying $x \neq y$, $\tilde{q}(x,y) > 0$ if and only if $q(y,x) > 0$;

(ii) For each $x \in S$, $\sum_{y\neq x} \tilde{q}(x,y) = \sum_{y\neq x} q(x,y)$.

In light of the homogeneous structure of $\mathbf{Q}$, we can choose $\tilde{\mathbf{Q}}$ so that it also exhibits a block-partitioned structure that is constructed using only three matrices $\tilde{\mathbf{A}}-1, \tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_1 \in \mathbb{R}^{d\times d}$, where for each $i, j \in \{1, 2, \ldots, d\}$ (where possibly $i = j$), and each $n \in \mathbb{Z}$,

$$\tilde{q}((n,i),(n-1,j)) = (\tilde{\mathbf{A}}_{-1})_{i,j}, \quad \tilde{q}((n,i),(n,j)) = (\tilde{\mathbf{A}}_0)_{i,j}, \quad \tilde{q}((n,i),(n+1,j)) = (\tilde{\mathbf{A}}_1)_{i,j}$$

and for any two integers n, m satisfying $|n-m| \geq 2$, $\tilde{q}((n,i),(m,j)) = 0$. for each $i, j \in \{1, 2, \ldots, d\}$.

We further associate with $\{\tilde{F}(t); t \geq 0\}$ the DTMC $\{\tilde{F}_n\}_{n \geq 0}$, where $\tilde{F}_0 := \tilde{F}(0)$, and for each integer $n \geq 1$, $\tilde{F}_n$ represents the state of $\{F(t); t \geq 0\}$ immediately after its nth transition time. We further associate with both $\{\tilde{F}(t); t \geq 0\}$ and $\{\tilde{F}_n\}_{n \geq 0}$ the following hitting-time random variables: for each subset $A \subset S$,

$$\tilde{\tau}_A := \inf\{t \geq 0 : F(t) \in A\}, \quad \tilde{\eta}_A := \inf\{n \geq 0 : F_n \in A\}$$

and for each state $x \in S$, we set $\tilde{\tau}_x := \tilde{\tau}_{\{x\}}$ and $\tilde{\eta}_x := \tilde{\eta}_{\{x\}}$.

Next, recall from [11] that for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{R}(\alpha)^{n-a})_{i,j} := \mathbb{E}_{(n,j)} \left[\mathbf{1}(\tilde{\eta}_{L_a} < \infty) \mathbf{1}(\tilde{F}(\tilde{\tau}_{L_a}) = (a,i)) e^{-\tilde{\tau}_{L_a}} \prod_{\ell=1}^{\tilde{\eta}_{L_a}} \frac{q(\tilde{F}_\ell, \tilde{F}_{\ell-1})}{\tilde{q}(\tilde{F}_{\ell-1}, \tilde{F}_\ell)} \right].$$

We can see from the Strong Markov property that

$$\begin{aligned}
(\mathbf{R}(\alpha)^{n-a})_{i,j} &= \mathbb{E}_{(n,j)} \left[\mathbf{1}(\tilde{\eta}_{L_{a,b}} < \infty) \mathbf{1}(\tilde{F}(\tilde{\tau}_{L_{a,b}}) = (a,i)) e^{-\tilde{\tau}_{L_{a,b}}} \prod_{\ell=1}^{\tilde{\eta}_{L_{a,b}}} \frac{q(\tilde{F}_\ell, \tilde{F}_{\ell-1})}{\tilde{q}(\tilde{F}_{\ell-1}, \tilde{F}_\ell)} \right] \\
&+ \sum_{k=1}^{M} \mathbb{E}_{(n,j)} \left[\mathbf{1}(\tilde{\eta}_{L_{a,b}} < \infty) \mathbf{1}(\tilde{F}(\tilde{\tau}_{L_{a,b}}) = (b,k)) e^{-\tilde{\tau}_{L_{a,b}}} \prod_{\ell=1}^{\tilde{\eta}_{L_{a,b}}} \frac{q(\tilde{F}_\ell, \tilde{F}_{\ell-1})}{\tilde{q}(\tilde{F}_{\ell-1}, \tilde{F}_\ell)} \right] (\mathbf{R}(\alpha)^{b-a})_{i,k} \\
&= (\mathbf{R}_{a,\{a,b\},n}(\alpha))_{i,j} + \sum_{k=1}^{M} (\mathbf{R}(\alpha)^{b-a})_{i,k} (\mathbf{R}_{b,\{a,b\},n}(\alpha))_{k,j}
\end{aligned}$$

which implies

$$\mathbf{R}(\alpha)^{n-a} = \mathbf{R}_{a,\{a,b\},n}(\alpha) + \mathbf{R}(\alpha)^{b-a} \mathbf{R}_{b,\{a,b\},n}(\alpha).$$

A similar argument reveals that

$$\mathbf{R}(\alpha)^{b-n} = \hat{\mathbf{R}}(\alpha)^{b-a} \mathbf{R}_{a,\{a,b\},n}(\alpha) + \hat{\mathbf{R}}_{b,\{a,b\},n}(\alpha)$$

which proves the claim. $\square$

We close this section by noting that the matrices $\mathbf{G}(\alpha)$ and $\hat{\mathbf{G}}(\alpha)$ can be calculated by using the iterative process explained in [10]. Once this is done, $\mathbf{R}(\alpha)$ and $\hat{\mathbf{R}}(\alpha)$ can be found by noting that

$$\mathbf{R}(\alpha) = \mathbf{A}_1 (\alpha \mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 \mathbf{G}(\alpha))^{-1}$$

and

$$\hat{\mathbf{R}}(\alpha) = \mathbf{A}_{-1} (\alpha \mathbf{I} - \mathbf{A}_0 - \mathbf{A}_{-1} \hat{\mathbf{G}}(\alpha))^{-1}.$$

These formulas are very well-known for the case where $\alpha = 0$: see e.g. Chapter 8 of [16].

3 Homogeneous QBD Processes with Finitely Many Levels

The matrices found in Lemmas 2.1 and 2.2 of the previous section can be used to study the time-dependent behavior of a homogeneous QBD process with finitely many levels. Suppose $\{Y(t); t \geq 0\}$ is a QBD process whose state space is given by S, where S is decomposed into a finite number of levels $L_0, L_1, \ldots, L_C$ for some integer $C \geq 1$, i.e.

$$S = \bigcup_{n=0}^{C} L_n.$$

We assume each level L_n is defined as

$$L_n := \{(n,1),(n,2),\ldots,(n,d)\}$$

for some fixed positive integer d. The transition rate matrix $\mathbf{Q} := [q(x,y)]_{x,y\in S}$ of $\{Y(t); t \geq 0\}$ can be expressed in block-partitioned form as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{-1} & \mathbf{A}_0 & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_{-1} & \mathbf{C}_0 \end{pmatrix}$$

where $\mathbf{0} \in \mathbb{R}^{d\times d}$ is the zero matrix, and $\mathbf{B}_0, \mathbf{C}_0, \mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1 \in \mathbb{R}^{d\times d}$ are structured so that $\mathbf{Q}$ satisfies the properties of a generator matrix associated with an irreducible, stable, and conservative continuous-time Markov chain: in other words, each off-diagonal element of $\mathbf{Q}$ is nonnegative, each diagonal element of $\mathbf{Q}$ is strictly negative and finite, and for each fixed row of $\mathbf{Q}$, the elements of that row always sum to zero. Readers should note the word 'stable' used here does not refer to positive recurrence, rather, it refers to the fact that each row sum of $\mathbf{Q}$ is zero: this terminology is commonly used in the literature on continuous-time Markov chains, see for instance the text of Anderson [1]. Having said this, due to S being finite and $\{Y(t); t \geq 0\}$ being irreducible, we may also conclude that $\{Y(t); t \geq 0\}$ is also 'stable' in the sense of positive recurrence. Note too that the number of elements in L_0 and L_C could possibly be different from d, but in the interest of readability we assume throughout that each level contains d states.

The block-partitioned structure exhibited above by $\mathbf{Q}$ corresponds to the way S is decomposed into levels, as the order the rows and columns of $\mathbf{Q}$ corresponds to the states of S being ordered lexicographically, meaning $(i_1, j_1) < (i_2, j_2)$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. Moreover, for each $i, j \in \{1, 2, \ldots, d\}$, where possibly $i = j$, we have (i)

$$q((0,i),(0,j)) = (\mathbf{B}_0)_{i,j}, \quad q((C,i),(C,j)) = (\mathbf{C}_0)_{i,j};$$

(ii) for each integer $n \in \{0, 1, \ldots, C-1\}$,

$$q((n,i),(n+1,j)) = (\mathbf{A}_1)_{i,j};$$

(iii) for each integer $n \in \{1, 2, \ldots, C\}$,

$$q((n,i),(n-1,j)) = (\mathbf{A}_{-1})_{i,j};$$

and finally (iv) for each integer $n \in \{1, 2, \ldots, C-1\}$,

$$q((n,i),(n,j)) = (\mathbf{A}_0)_{i,j}.$$

We will also need to make use of hitting-time random variables associated with $\{Y(t); t \geq 0\}$. For each subset A of S, we define

$$\tau_A := \inf\{t \geq 0 : Y(t-) \neq Y(t) \in A\}$$

where for each $t > 0$, $Y(t-) := \lim_{s\uparrow t} Y(s)$ is the left-hand-limit of Y at t.

3.1 Distribution of the time it takes to reach a level

For each $m, n \in \{0, 1, \ldots, C\}$ satisfying $m > n$, we define the matrix $\mathbf{G}_{m,n}(\alpha)$ as follows: for each $i, j \in \{1, 2, \ldots, M\}$, we have

$$(\mathbf{G}_{m,n}(\alpha))_{i,j} := \mathbb{E}_{(m,i)}[e^{-\alpha\tau_{L_n}}\mathbf{1}(Y(\tau_{L_n}) = (n, j))].$$

Similarly, for each $m, n \in \{0, 1, \ldots, C\}$ satisfying $m < n$, we define the matrix $\hat{\mathbf{G}}_{m,n}(\alpha)$ as

$$(\hat{\mathbf{G}}_{m,n}(\alpha))_{i,j} := \mathbb{E}_{(m,i)}[e^{-\alpha\tau_{L_n}}\mathbf{1}(Y(\tau_{L_n}) = (n, j))].$$

We further define, for $0 \leq a < n < b \leq C$, the matrices $\mathbf{G}_{n,a,b}(\alpha)$ and $\hat{\mathbf{G}}_{n,b,a}(\alpha)$ as

$$(\mathbf{G}_{n,a,b}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_{a,b}}}\mathbf{1}(Y(\tau_{L_{a,b}}) = (a, j))], \quad (\hat{\mathbf{G}}_{n,b,a}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}[e^{-\alpha\tau_{L_{a,b}}}\mathbf{1}(Y(\tau_{L_{a,b}}) = (b, j))]$$

where just as in the previous section, $L_{a,b} := L_a \cup L_b$. It is easy to see that the matrices $\mathbf{G}_{n,a,b}(\alpha)$ and $\hat{\mathbf{G}}_{n,b,a}(\alpha)$ are equal to the matrices we defined in the previous section.

Our next proposition provides us with the matrices needed in order to derive the Laplace-Stieltjes transform of the amount of time it takes $\{Y(t); t \geq 0\}$ to move from one fixed level to another fixed level.

Proposition 3.1 *The matrices* $\{\mathbf{G}_{m,n}(\alpha)\}_{0 \leq m,n \leq C; m \neq n}$ *are as follows:* (i) *first,*

$$\hat{\mathbf{G}}_{0,1}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1, \quad \mathbf{G}_{C,C-1}(\alpha) = (\alpha\mathbf{I} - \mathbf{C}_0)^{-1}\mathbf{A}_{-1}. \tag{5}$$

(ii) *For each integer* $n \geq 2$,

$$\hat{\mathbf{G}}_{0,n}(\alpha) = [\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)]^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha). \tag{6}$$

(iii) *For each integer* $n \leq C - 2$,

$$\mathbf{G}_{C,n}(\alpha) = \left[\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n}(\alpha)\right]^{-1}\mathbf{A}_{-1}\mathbf{G}_{C-1,n,C}(\alpha). \tag{7}$$

(iv) *For each integer* $m \in \{1, 2, \ldots, C-1\}$ *and each integer* $n \in \{m+1, m+2, \ldots, C\}$,

$$\hat{\mathbf{G}}_{m,n}(\alpha) = \hat{\mathbf{G}}_{m,n,0}(\alpha) + \mathbf{G}_{m,0,n}(\alpha)[\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)]^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha). \tag{8}$$

(v) *Finally, for each integer* $m \in \{1, 2, \ldots, C-1\}$ *and each integer* $n \in \{0, 1, \ldots, m-1\}$,

$$\mathbf{G}_{m,n}(\alpha) = \mathbf{G}_{m,n,C}(\alpha) + \hat{\mathbf{G}}_{m,C,n}(\alpha)\left[\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n}(\alpha)\right]^{-1}\mathbf{A}_{-1}\mathbf{G}_{C-1,n,C}(\alpha). \tag{9}$$

Proof We first show that the matrices $(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha))$ are invertible for each integer $n \geq 2$, and the matrices $(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n}(\alpha))$ are invertible for each $n \leq C - 2$. Given a subset $A \subset \{0, 1, 2, \ldots, C\}$, we define for each $m \in A^c$ the matrix $\mathbf{N}_{m,A}(\alpha)$, defined as

$$(\mathbf{N}_{m,A}(\alpha))_{i,j} := \mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_A}} e^{-\alpha t}\mathbf{1}(Y(t) = (m, j))\right],$$

where we recall that for each subset $A \subset \{0, 1, \ldots, C\}$, $L_A := \cup_{n \in A} L_n$.

Fixing $i, j \in \{1, 2, \ldots, M\}$, we observe through a first-step analysis argument that for each integer $n \in \{2, 3, \ldots, C\}$,

$$\begin{aligned}\mathbb{E}_{(0,i)}\left[\int_0^{\tau_{L_n}} e^{-\alpha t}\mathbf{1}(Y(t) = (0, j))dt\right] &= \frac{\mathbf{1}(i = j)}{-(\mathbf{B}_0)_{i,i} + \alpha} \\ &+ \sum_{k \neq i} \frac{(\mathbf{B}_0)_{i,k}}{-(\mathbf{B}_0)_{i,i} + \alpha}(\mathbf{N}_{0,\{n\}}(\alpha))_{k,j} \\ &+ \sum_{k} \frac{(\mathbf{A}_1)_{i,k}}{-(\mathbf{B}_0)_{i,i} + \alpha}\mathbb{E}_{(1,i)}\left[\int_0^{\tau_{L_n}} e^{-\alpha t}\mathbf{1}(Y(t) = (0, j))dt\right]\end{aligned}$$

and after applying the Strong Markov property to the remaining expectations and rewriting the equations in terms of matrices, we get

$$\alpha\mathbf{N}_{0,\{n\}}(\alpha) = \mathbf{I} + \mathbf{B}_0\mathbf{N}_{0,\{n\}}(\alpha) + \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\mathbf{N}_{0,\{n\}}(\alpha)$$

which implies

$$(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha))\mathbf{N}_{0,\{n\}}(\alpha) = \mathbf{I}$$

thus proving $(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha))$ is invertible. A similar argument can be used to show that $(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\mathbf{G}_{C-1,C,n}(\alpha))$ is invertible for $n \in \{0, 1, \ldots, C-2\}$, if we replace $\mathbf{N}_{0,\{n\}}(\alpha)$ with the matrix $\mathbf{N}_{C,\{n\}}(\alpha)$.

It remains to establish statements (5)-(9). The first equality found in (5) can be proven with a first-step analysis argument: for each $i, j \in \{1, 2, \ldots, M\}$, where possibly $i = j$, we have

$$(\hat{\mathbf{G}}_{0,1}(\alpha)))_{i,j} = \sum_{k \neq i} \frac{(\mathbf{B}_0)_{i,k}}{-(\mathbf{B}_0)_{i,i} + \alpha}(\hat{\mathbf{G}}_{0,1}(\alpha))_{k,j} + \frac{(\mathbf{A}_1)_{i,j}}{-(\mathbf{B}_0)_{i,i} + \alpha}$$

and these equations can alternatively be expressed in matrix form as

$$\hat{\mathbf{G}}_{0,1}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1.$$

The other equality found in statement (5) follows from an analogous argument.

We next prove statement (6): again, a first-step analysis argument can be used to show that for each integer $n \geq 2$,

$$\hat{\mathbf{G}}_{0,n}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n}(\alpha).$$

Furthermore,

$$\hat{\mathbf{G}}_{1,n}(\alpha) = \hat{\mathbf{G}}_{1,n,0}(\alpha) + \mathbf{G}_{1,0,n}(\alpha)\hat{\mathbf{G}}_{0,n}(\alpha).$$

Hence,

$$\hat{\mathbf{G}}_{0,n}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha) + (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\hat{\mathbf{G}}_{0,n}(\alpha)$$

and solving for the single unknown matrix gives

$$\begin{aligned}\hat{\mathbf{G}}_{0,n}(\alpha) &= \left[\mathbf{I} - (\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\right]^{-1}(\alpha\mathbf{I} - \mathbf{B}_0)^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha) \\ &= \left[\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n}(\alpha)\right]^{-1}\mathbf{A}_1\hat{\mathbf{G}}_{1,n,0}(\alpha)\end{aligned}$$

proving (6). A similar argument can be used to establish (7).

Statement (8) follows from (6), once we notice that for $0 < m < n \leq C$,

$$\hat{\mathbf{G}}_{m,n}(\alpha) = \hat{\mathbf{G}}_{m,n,0}(\alpha) + \mathbf{G}_{m,0,n}(\alpha)\hat{\mathbf{G}}_{0,n}(\alpha)$$

and a similar argument can be used to show that (9) follows from (7). □

3.2 The Laplace Transforms of the Transition Functions

Together, Lemmas 2.1, 2.2, and Proposition 3.1 can be used to derive what appear to be new, computable expressions for the Laplace transforms of the transition functions of $\{Y(t); t \geq 0\}$. We assume throughout (and without loss of generality) that $Y(0) = (n_0, i_0)$ with probability one for some state $(n_0, i_0) \in S$. For each state $(n, j) \in S$, we define the transition function $p_{(n_0,i_0),(n,j)} : [0, \infty) \to [0, 1]$ as

$$p_{(n_0,i_0),(n,j)}(t) := \mathbb{P}(Y(t) = (n, j) \mid Y(0) = (n_0, i_0)), \quad t \geq 0.$$

Associated with $p_{(n_0,i_0),(n,j)}(t)$ is its Laplace transform $\pi_{(n_0,i_0),(n,j)}$ which is defined on $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$ as

$$\pi_{(n_0,i_0),(n,j)}(\alpha) := \int_0^\infty e^{-\alpha t} p_{(n_0,i_0),(n,j)}(t)dt, \quad \alpha \in \mathbb{C}_+.$$

Our next result, Theorem 3.1, is stated in [10], and is a Laplace transform interpretation of an unlabeled result found at the top of page 124 of [16].

Theorem 3.1 *Suppose T and D are two disjoint subset of S. Then for each $x \in T$, $y \in D$,*

$$\pi_{x,y}(\alpha) = \sum_{z \in T} \pi_{x,z}(\alpha)(q(z) + \alpha)\mathbb{E}_z \left[\int_0^{\tau_T} e^{-\alpha t} \mathbf{1}(Y(t) = y)dt \right], \quad \alpha \in \mathbb{C}_+$$

We can use Theorem 3.1 to establish a result that can be used to find the Laplace transform of the transitions functions of $\{Y(t); t \geq 0\}$. Since our results will be in matrix form, we define

$$\boldsymbol{\pi}_n(\alpha) := [\pi_{(n_0,i_0),(n,1)}(\alpha), \pi_{(n_0,i_0),(n,2)}(\alpha), \ldots, \pi_{(n,M)}(\alpha)], \quad \alpha \in \mathbb{C}_+.$$

We suppress the initial state (n_0, i_0) when we write $\boldsymbol{\pi}_n(\alpha)$, but readers should understand that these vectors depend on the initial state.

The next result, Theorem 3.2, provides an expression for the Laplace transforms of the transition functions of $\{Y(t); t \geq 0\}$ that is highly analogous to the expressions found in [9] for the stationary distribution of $\{Y(t); t \geq 0\}$, for the case where $\{Y(t); t \geq 0\}$ is non-null recurrent. Throughout, the vector $\mathbf{e}_i$ denotes the ith basis vector in $\mathbb{R}^{1\times d}$, where the ith component of $\mathbf{e}_i$ is equal to one, and all of its other components are equal to zero.

Theorem 3.2 *The Laplace transforms of the transition functions of $\{Y(t); t \geq 0\}$ are as follows.*
(i) If $n_0 = 0$, we see that for $1 \leq n \leq C - 1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) &= \left[\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1} - \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\hat{\mathbf{R}}(\alpha)^C\right]\mathbf{R}(\alpha)^n \\ &\quad + \left[-\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1}\mathbf{R}(\alpha)^C + \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\right]\hat{\mathbf{R}}(\alpha)^{C-n}. \quad (10)\end{aligned}$$

The vectors $\boldsymbol{\pi}_0(\alpha)$ and $\boldsymbol{\pi}_C(\alpha)$ satisfy

$$\boldsymbol{\pi}_C(\alpha) = \boldsymbol{\pi}_0(\alpha)\mathbf{A}_1\hat{\mathbf{G}}_{1,C,0}(\alpha)(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,0}(\alpha))^{-1} \quad (11)$$

and

$$\boldsymbol{\pi}_0(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0}(\alpha))^{-1}. \quad (12)$$

(ii) If $n_0 = C$, we see that for $1 \leq n \leq C - 1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) &= \left[\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1} - \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\hat{\mathbf{R}}(\alpha)^C\right]\mathbf{R}(\alpha)^n \\ &\quad + \left[-\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^C\hat{\mathbf{R}}(\alpha)^C]^{-1}\mathbf{R}(\alpha)^C + \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^C\mathbf{R}(\alpha)^C]^{-1}\right]\hat{\mathbf{R}}(\alpha)^{C-n}. \quad (13)\end{aligned}$$

The vectors $\boldsymbol{\pi}_0(\alpha)$ and $\boldsymbol{\pi}_C(\alpha)$ satisfy

$$\boldsymbol{\pi}_0(\alpha) = \boldsymbol{\pi}_C(\alpha)\mathbf{A}_{-1}\mathbf{G}_{C-1,0,C}(\alpha)(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,C}(\alpha))^{-1} \quad (14)$$

and

$$\boldsymbol{\pi}_C(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C}(\alpha))^{-1} \quad (15)$$

(*iii*) *Finally, suppose* $n_0 \in \{1, 2, \ldots, C-1\}$. *For* $1 \le n \le n_0 - 1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) = &\Big[\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{n_0}\hat{\mathbf{R}}(\alpha)^{n_0}]^{-1} - \boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{n_0}\mathbf{R}(\alpha)^{n_0}]^{-1}\hat{\mathbf{R}}(\alpha)^{n_0}\Big]\mathbf{R}(\alpha)^n \\ &+ \Big[-\boldsymbol{\pi}_0(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{n_0}\hat{\mathbf{R}}(\alpha)^{n_0}]^{-1}\mathbf{R}(\alpha)^{n_0} + \boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{n_0}\mathbf{R}(\alpha)^{n_0}]^{-1}\Big]\hat{\mathbf{R}}(\alpha)^{n_0-n}.\end{aligned} \tag{16}$$

For $n_0 + 1 \le n \le C - 1$,

$$\begin{aligned}\boldsymbol{\pi}_n(\alpha) = &\Big[\boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{C-n_0}\hat{\mathbf{R}}(\alpha)^{C-n_0}]^{-1} - \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{C-n_0}\mathbf{R}(\alpha)^{C-n_0}]^{-1}\hat{\mathbf{R}}(\alpha)^{C-n_0}\Big]\mathbf{R}(\alpha)^{n-n_0} \\ &+ \Big[-\boldsymbol{\pi}_{n_0}(\alpha)[\mathbf{I} - \mathbf{R}(\alpha)^{C-n_0}\hat{\mathbf{R}}(\alpha)^{C-n_0}]^{-1}\mathbf{R}(\alpha)^{C-n_0} + \boldsymbol{\pi}_C(\alpha)[\mathbf{I} - \hat{\mathbf{R}}(\alpha)^{C-n_0}\mathbf{R}(\alpha)^{C-n_0}]^{-1}\Big]\hat{\mathbf{R}}(\alpha)^{C-n}.\end{aligned} \tag{17}$$

The vectors $\boldsymbol{\pi}_0(\alpha)$, $\boldsymbol{\pi}_C(\alpha)$, *and* $\boldsymbol{\pi}_{n_0}(\alpha)$ *satisfy*

$$\boldsymbol{\pi}_0(\alpha) = \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{A}_{-1}\mathbf{G}_{n_0-1,0,n_0}(\alpha)(\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n_0}(\alpha))^{-1} \tag{18}$$

$$\boldsymbol{\pi}_C(\alpha) = \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{A}_1\hat{\mathbf{G}}_{n_0+1,C,n_0}(\alpha)(\alpha\mathbf{I} - \mathbf{C}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{C-1,C,n_0}(\alpha))^{-1} \tag{19}$$

and

$$\boldsymbol{\pi}_{n_0}(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_{-1}\hat{\mathbf{G}}_{n_0-1,n_0}(\alpha) - \mathbf{A}_1\mathbf{G}_{n_0+1,n_0}(\alpha))^{-1}. \tag{20}$$

Proof We begin the proof by first setting up some additional notation. For each subset $A \subset \{0, 1, 2, \ldots, C\}$, each $m \in A$, and each $n \in A^c$, define the matrix $\mathbf{R}^{(0,C)}_{m,A,n}(\alpha)$ as follows: for each $i, j \in \{1, 2, \ldots, d\}$,

$$(\mathbf{R}^{(0,C)}_{m,A,n}(\alpha))_{i,j} := (q((m,i)) + \alpha)\mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_A}} e^{-\alpha t}\mathbf{1}(Y(t) = (n,j))dt\right].$$

It is obvious from the transition structure of both $\{F(t); t \ge 0\}$ and $\{Y(t); t \ge 0\}$ that for $0 \le a < n < b \le C$,

$$\mathbf{R}^{(0,C)}_{a,\{a,b\},n}(\alpha) = \mathbf{R}_{a,\{a,b\},n}(\alpha)$$

and precisely the same can be said for $\mathbf{R}_{b,\{a,b\},n}(\alpha)$ and $\mathbf{R}^{(0,C)}_{b,\{a,b\},n}(\alpha)$.

We focus on case (iii) by establishing the validity of (16), (17), (18), (19), and (20), which all correspond to the case where $n_0 \in \{1, 2, \ldots, C-1\}$. Fix such an n_0: observe first that for $0 < n < n_0$, an application of Theorem 3.1, under the choice $T = L_0 \cup L_{n_0}$ yields

$$\boldsymbol{\pi}_n(\alpha) = \boldsymbol{\pi}_0(\alpha)\mathbf{R}_{0,\{0,n_0\},n}(\alpha) + \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{R}_{n_0,\{0,n_0\},n}(\alpha) \tag{21}$$

and applying both (3) and (4) to (21) yields (17). A similar argument can be used to establish (17) for $n \in \{n_0 + 1, \ldots, C-1\}$, where in that case we apply Theorem 3.1 while instead choosing $T = L_{n_0} \cup L_C$, then again applying Lemma 2.2.

The next step is to establish (18). Applying Theorem 3.1 while choosing $T = L_{n_0}$ gives

$$\boldsymbol{\pi}_0(\alpha) = \boldsymbol{\pi}_{n_0}(\alpha)\mathbf{R}^{(0,C)}_{n_0,\{n_0\},0}(\alpha). \tag{22}$$

Conditioning on the first jump and using the strong Markov property we see that

$$\mathbf{R}^{(0,C)}_{n_0,\{n_0\},0}(\alpha) = \mathbf{A}_{-1}\mathbf{G}_{n_0-1,0,n_0}(\alpha)\mathbf{N}_{0,\{n_0\}}(\alpha) \tag{23}$$

where we recall that for each integer $n \neq n_0$, the matrix $\mathbf{N}_{n,\{n_0\}}(\alpha)$ is defined as

$$(\mathbf{N}_{n,\{n_0\}}(\alpha))_{i,j} := \mathbb{E}_{(n,i)}\left[\int_0^{\tau_{L_{n_0}}} e^{-\alpha t}\mathbf{1}(Y(t) = (n,j))dt\right].$$

A first-step analysis argument can be used to show that

$$\mathbf{N}_{0,\{n_0\}}(\alpha) = (\alpha\mathbf{I} - \mathbf{B}_0 - \mathbf{A}_1\mathbf{G}_{1,0,n_0}(\alpha))^{-1}. \tag{24}$$

Plugging (24) into (23), and plugging that into (22) yields (18), and the same type of reasoning used to establish (18) can be used to establish (19).

It remains to derive (20). For each $m, n \in \{0, 1, 2, \ldots, C\}$, where possibly $m = n$, we define the matrix $\Pi_{m,n}(\alpha)$ as

$$\Pi_{m,n}(\alpha) := [\pi_{(m,i),(n,j)}(\alpha)]_{1 \leq i,j \leq M}.$$

From the Forward equations associated with $\{Y(t); t \geq 0\}$, we see that

$$\alpha\Pi_{n_0,n_0}(\alpha) - \mathbf{I} = \Pi_{n_0,n_0-1}(\alpha)\mathbf{A}_1 + \Pi_{n_0,n_0}(\alpha)\mathbf{A}_0 + \Pi_{n_0,n_0+1}(\alpha)\mathbf{A}_{-1}$$

which yields

$$\Pi_{0,0}(\alpha)(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0-1}(\alpha)\mathbf{A}_1 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0+1}(\alpha)\mathbf{A}_{-1}) = \mathbf{I}$$

from which we get

$$\boldsymbol{\pi}_0(\alpha) = \mathbf{e}_{i_0}(\alpha\mathbf{I} - \mathbf{A}_0 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0-1}(\alpha)\mathbf{A}_1 - \mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0+1}(\alpha)\mathbf{A}_{-1})^{-1}. \tag{25}$$

We now claim that

$$\mathbf{G}_{n_0+1,n_0}(\alpha) = \mathbf{N}_{n_0+1,\{n_0\}}(\alpha)\mathbf{A}_{-1}.$$

One way to show this involves a technique found in Chapter 9 of Brémaud [2], where a CTMC is thought of as being governed entirely by a countable collection of independent homogeneous Poisson processes. In our case, suppose $\{Y(t); t \geq 0\}$ is governed by the collection of Poisson processes $\{N_{x,y}(t)\}_{x,y \in S, x \neq y}$. Define $\theta(t) = 0$ if L_{n_0} has not been visited yet by time t and 1 otherwise. Then, if $Y(0) = (n_0 + 1, i)$,

$$\mathbf{1}(Y(\tau_{L_{n_0}}) = (n_0, j))e^{-\alpha\tau_{L_0}} = \sum_k \int_0^\infty e^{-\alpha t}\mathbf{1}(Y(t-) = (n_0+1, k), \theta(t-) = 0)N_{(n_0+1,k),(n_0,j)}(dt).$$

Taking the expectation of both sides while applying the Campbell-Mecke formula to the right-hand side gives

$$\mathbb{E}_{(n_0+1,i)}[\mathbf{1}(Y(\tau_{L_{n_0}}) = (n_0, j))e^{-\alpha\tau_{L_0}}] = \sum_k \mathbb{E}_{(n_0+1,i)}\left[\int_0^{\tau_{L_{n_0}}} e^{-\alpha t}\mathbf{1}(Y(t) = (n_0+1, k))dt\right](\mathbf{A}_{-1})_{k,j}$$

or, in matrix form,

$$\mathbf{G}_{n_0+1,n_0}(\alpha) = \mathbf{N}_{n_0+1,\{n_0\}}(\alpha)\mathbf{A}_{-1}.$$

Thus we now have

$$\mathbf{R}^{(0,C)}_{n_0,\{n_0\},n_0+1}(\alpha)\mathbf{A}_{-1} = \mathbf{A}_1\mathbf{N}_{n_0+1,\{n_0\}}(\alpha)\mathbf{A}_{-1} = \mathbf{A}_1\mathbf{G}_{n_0+1,n_0}(\alpha). \tag{26}$$

Analogously, we can also show

$$\hat{\mathbf{R}}^{(0,C)}_{n_0,\{n_0\},n_0-1}(\alpha)\mathbf{A}_1 = \mathbf{A}_{-1}\mathbf{N}_{n_0-1,\{n_0\}}(\alpha)\mathbf{A}_1 = \mathbf{A}_{-1}\hat{\mathbf{G}}_{n_0-1,n_0}(\alpha). \tag{27}$$

Substituting equations (26) and (27) into equation (25) gives (20), which concludes the proof of the theorem. □

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