On the study of the running maximum and minimum level of level-dependent Quasi-Birth-Death processes and related models

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Abstract

We present a study of the joint distribution of both the state of a level-dependent Quasi-Birth-Death (QBD) process, as well as its associated running maximum level, at a fixed time t: more specifically, we derive expressions for the Laplace transforms of transition functions that contain this information, and the expressions we derive contain familiar constructs from the classical theory of QBD processes. We also explain how our methods extend to the study of level-dependent Markov processes of M/G/1-type.

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1 Introduction and Preliminary Results

Given a real-valued stochastic process $\{X(t); t \geq 0\}$, we can define both the *running maximum* process $\{\overline{X}(t); t \geq 0\}$ and the *running minimum* process $\{\underline{X}(t); t \geq 0\}$, where for each $t \geq 0$,

$$\overline{X}(t) := \sup_{s \in [0,t]} X(s), \qquad \underline{X}(t) := \inf_{s \in [0,t]} X(s).$$

The marginal distributions of these processes are very tractable when $\{X(t); t \geq 0\}$ represents Brownian motion, and they are also well-known to play a prominent role in the theory of Lévy processes: readers seeking an introduction to Lévy processes are referred to Kyprianou [9].

In the recent work of Mandjes and Taylor [11], the authors present a recursive procedure that can be used to calculate the joint distribution of both the state (which tracks level and phase) of a level-dependent Quasi-Birth-Death (QBD) process and its running maximum level, at an independent exponential time: once these distributions can be calculated efficiently, Erlangization can be used to further study, numerically, the joint distribution of the running maximum level, the level, and the phase at each fixed time t. The results contained in [11] were derived 'from scratch' by making clever use of first-step analysis and censoring arguments, as well as sample-path properties satisfied by level-dependent QBD processes. Our objective is to build further on the work of [11] by showing how alternative formulas can be derived in an arguably more straightforward manner from theory that has been developed in the matrix-analytic literature. In fact, not only will we analyze level-dependent QBD processes, we will also explain how our results and ideas apply to level-dependent Markov processes of M/G/1-type, assuming of course that we replace the running maximum level process with a running minimum level process.

An important ingredient needed in our analysis is a formula that can be found at the top of page 124 of Latouche and Ramaswami [10], which we now describe in reasonable detail. Suppose

$\{Y(t); t \geq 0\}$ is a continuous-time Markov chain (CTMC) having state space S and generator (transition rate matrix) $\mathbf{Q} := [q(x, y)]_{x,y \in S}$, where for each $x \in S$,

$$q(x) := -q(x, x) \geq 0$$

denotes the sojourn rate associated with each exponential sojourn spent in state x by $\{Y(t); t \geq 0\}$. We assume throughout that each CTMC we study satisfies the property that $q(x) < \infty$ for each $x \in S$.

Further associated with $\{Y(t); t \geq 0\}$ is a collection of transition functions $\{p_{x,y}\}_{x,y \in S}$, where for each $x, y \in S$,

$$p_{x,y}(t) := \mathbb{P}_x(Y(t) = y), \qquad t \geq 0$$

where $\mathbb{P}_x(\cdot)$ represents a conditional probability, given $Y(0) = x$. Each transition function $p_{x,y}$ has associated with it a Laplace transform $\pi_{x,y} : \mathbb{C}_+ \to \mathbb{C}$, which is defined on $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$—the set of all complex numbers having positive real part—as

$$\pi_{x,y}(\alpha) := \int_0^\infty e^{-\alpha t} p_{x,y}(t) dt, \qquad \alpha \in \mathbb{C}_+.$$

Readers should recall that two continuous functions defined on $[0, \infty)$ are equal if and only if their Laplace transforms are equal on $\mathbb{C}_+$ (in fact the functions are equal if and only if their Laplace transforms are equal on $[0, \infty)$) and once we can numerically calculate a Laplace transform at each point in $\mathbb{C}_+$, we can use one of many numerical transform inversion algorithms, such as that found in [1], to calculate the underlying continuous function at various points of $[0, \infty)$.

For each subset $T \subset S$, we define

$$\tau_T := \inf\{t \geq 0 : Y(t-) \neq Y(t) \in T\}$$

which represents the first time $\{Y(t); t \geq 0\}$ makes a transition to a state contained in T. Readers should note that $\tau_T > 0$ with probability one, even if $X(0) \in T$, as τ_T represents the first time the chain makes a transition to a state in T, which could have been made from a state $x \in T$ if $X(0) = x$.

Theorem 1.1 (*page 124 of Latouche and Ramaswami*) *Suppose T is a nonempty subset of S, where $T \neq S$. Then for each $x \in T^c$, and each $y \in T$,*

$$p_{x,y}(t) = \sum_{z \in T^c} \sum_{w \in T} \int_0^t p_{x,z}(s) q(z, w) \mathbb{P}_w(Y(t-s) = y, \tau_{T^c} > t - s) ds. \tag{1}$$

While this result is obviously known, in [10] the formula appears to be given only with the intention of using it as a tool for deriving the stationary distribution of QBD processes, but we feel that this result deserves its own theorem. The authors of [10] appear to establish the result with a Markov renewal argument, but here is an alternative argument that follows from ideas found in [6].

Proof One way to derive Theorem 1.1 involves using the framework from Chapter 9 of Brémaud [2], where a CTMC is thought of as being governed entirely by a countable collection of independent, homogeneous Poisson processes.

Here is a rough sketch of the construction: for each $x, y \in S$ where $x \neq y$, we construct a Poisson process $\{N_{x,y}(t); t \geq 0\}$ with rate $q(x, y)$. Setting now $Y(0) = y_0$—an arbitrarily chosen state—we define the first transition time T_1 of $\{Y(t); t \geq 0\}$ as

$$T_1 := \inf_{y \in S} \inf\{t \geq 0 : N_{y_0,y}(t) = 1\}$$

and we set $Y(t) = y_0$ for $0 \leq t < T_1$, with $Y(T_1) = y_1$ for that state y_1 that attains the infimum (such a state exists with probability one). Next, given $y_1 = Y(T_1)$, set

$$T_2 := \inf_{y \in S} \inf\{t \geq 0 : N_{y_1,y}(t + T_1) - N_{y_1,y}(T_1) = 1\}$$

and again, define $Y(t) = y_1$ for $T_1 \leq t < T_2$, and set $Y(T_2) = y_2$ where y_2 is the state that attains the infimum. From here, one can define $\{Y(t); t \geq 0\}$ inductively over the entire line: note that it is possible for $\{Y(t); t \geq 0\}$ to have infinitely many transitions in a finite time interval, meaning

$$T_\infty := \lim_{n \to \infty} T_n < \infty$$

and in this case we construct an extra 'cemetery state' ∂, and assume the process stays at this cemetery state from the explosion time onward. Readers should find it clear, at least on an intuitive level, that $\{Y(t); t \geq 0\}$ is a CTMC with transition rate matrix $\mathbf{Q}$, but we refer those interested in seeing a rigorous description of this procedure to Chapter 9, Sections 1 and 2 of [2].

Thinking of $\{Y(t); t \geq 0\}$ in this manner, we can observe that for each $x \in T^c$ and each $y \in T$, if $Y(0) = x$ we have

$$\mathbf{1}(Y(t) = y) = \sum_{z \in T^c} \sum_{w \in T} \int_0^t \mathbf{1}(Y(s-) = z, \tau_{T^c}(s) > t, Y(t) = y) N_{z,w}(ds)$$

where $Y(s-)$ is the left-hand-limit of Y at s, and for each $C \subset S$,

$$\tau_C(s) := \inf\{t \geq s : Y(t-) \neq Y(t) \in C\}.$$

Taking the expectation of both sides, while further applying the Campbell-Mecke formula to the right-hand-side, as is done in [6], gives

$$\mathbb{P}_x(Y(t) = y) = \sum_{z \in T^c} \sum_{w \in T} \int_0^t \mathbb{P}_x(Y(s) = z) q(z, w) \mathbb{P}_w(\tau_{T^c} > t - s, Y(t-s) = y) ds$$

which proves the claim. □

Remark It is also possible to establish Theorem 1.1 via the *random-product technique*. Even though the random-product technique requires less of a technical background in measure-theoretic probability, when using this technique one has to specially treat both absorbing states, as well as states that cannot be reached from any state (meaning the only way the CTMC can visit this state is if it starts there). Such states may appear in a few places of our analysis, so we decided to motivate Theorem 1.1 with the line of reasoning given in [6], which uses the point process framework of [2].

The next result is a corollary of Theorem 1.1.

Corollary 1.1 *Fix a nonempty subset $T \subset S$ where $T \neq S$. Then for each $x \in T^c$, and each $y \in T$,*

$$\pi_{x,y}(\alpha) = \sum_{z \in T^c} \pi_{x,z}(\alpha)(q(z) + \alpha) \mathbb{E}_z \left[\int_0^{\tau_{T^c}} e^{-\alpha t} \mathbf{1}(Y(t) = y) dt \right], \qquad \alpha \in \mathbb{C}_+. \tag{2}$$

Proof This result follows from Theorem 1.1: simply multiply both sides of (1) by $e^{-\alpha t}$, integrate with respect to t over $[0, \infty)$, and apply Fubini's Theorem. □
Equation (2) can alternatively be stated as

$$\pi_{x,y}(\alpha) = \sum_{z \in T^c} \pi_{x,z}(\alpha) \sum_{w \in T} q(z, w) \mathbb{E}_w \left[\int_0^{\tau_{T^c}} e^{-\alpha t} \mathbf{1}(Y(t) = y) dt \right], \qquad \alpha \in \mathbb{C}_+. \tag{3}$$

We will often find it useful to state Equation (2) in this manner.

2 Level-Dependent QBD Processes

Suppose $\{Y(t); t \geq 0\}$ is a level-dependent QBD process, whose state space S is expressed in terms of a countable union of *levels*:

$$S := \bigcup_{n=0}^{\infty} L_n$$

where, for each integer $n \geq 0$,

$$L_n := \{(n,1),(n,2),\ldots,(n,d_n-1),(n,d_n)\}$$

with d_n being a fixed positive integer that is allowed to vary with n. Given the structure of S, it helps, for each $t \geq 0$, to express $Y(t)$ as

$$Y(t) = (X(t), J(t))$$

for each real $t \geq 0$, where $X(t)$ denotes the current level of the process—meaning $X(t) = n$ if and only if $Y(t) \in L_n$—and $J(t)$ represents the current *phase* of the process. We follow the notation scheme from [11] by letting $\mathbf{Q}$ denote the transition rate matrix of $\{Y(t); t \geq 0\}$, where the rows and columns of $\mathbf{Q}$ are ordered in a manner that corresponds to S being ordered lexicographically, so that

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}^{(0)} & \Lambda^{(0)} & \mathbf{0}_{d_0\times d_2} & \mathbf{0}_{d_0\times d_3} & \mathbf{0}_{d_0\times d_4} & \cdots \\ \mathcal{M}^{(1)} & \mathbf{Q}^{(1)} & \Lambda^{(1)} & \mathbf{0}_{d_1\times d_3} & \mathbf{0}_{d_1\times d_4} & \cdots \\ \mathbf{0}_{d_2\times d_0} & \mathcal{M}^{(2)} & \mathbf{Q}^{(2)} & \Lambda^{(2)} & \mathbf{0}_{d_2\times d_4} & \cdots \\ \mathbf{0}_{d_3\times d_0} & \mathbf{0}_{d_3\times d_1} & \mathcal{M}^{(3)} & \mathbf{Q}^{(3)} & \Lambda^{(3)} & \ddots \\ \mathbf{0}_{d_4\times d_0} & \mathbf{0}_{d_4\times d_1} & \mathbf{0}_{d_4\times d_2} & \mathcal{M}^{(4)} & \mathbf{Q}^{(4)} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

where $\mathbf{0}_{m\times n}$ represents the zero matrix with m rows and n columns.

From this description of $\mathbf{Q}$, we can see that the dimensions of $\mathbf{Q}^{(0)}$ and $\Lambda^{(0)}$ are $d_0 \times d_0$ and $d_0 \times d_1$, respectively, while for each integer $n \geq 1$, the dimensions of $\mathcal{M}^{(n)}$, $\mathbf{Q}^{(n)}$, and $\Lambda^{(n)}$ are $d_n \times d_{n-1}$, $d_n \times d_n$, and $d_n \times d_{n+1}$, respectively. Each matrix $\Lambda^{(n)}$ contains transition rates corresponding to transitions made from a state in L_n to a state in L_{n+1}, while each matrix $\mathcal{M}^{(n)}$ contains transition rates corresponding to transitions made from a state in L_n to a state in L_{n-1}. In the interest of avoiding 'nuisance states', we assume throughout that each state $x \in S$ satisfies the following condition: there exist two states $y, z \in S$ (which may depend on x) such that $q(x,y) > 0$ and $q(z,x) > 0$. This is a much more general condition than irreducibility, as we are assuming that $\{Y(t); t \geq 0\}$ has no absorbing states, nor are there states that cannot be reached in one step from any other state in S. This simple assumption will allow us to apply the random-product technique featured in [3, 4, 5] without further comment. Readers should note that in [11], the authors assume the structure of $\mathbf{Q}$ is such that $\{Y(t); t \geq 0\}$ is an irreducible CTMC, which in itself is a harmless assumption to make.

A very important family of matrices associated with $\{Y(t); t \geq 0\}$ is the family of '$\mathbf{R}$-matrices' $\{\mathbf{R}_{k+1,k}(\alpha)\}_{k\geq 0}$, where for each integer $k \geq 0$,

$$(\mathbf{R}_{k+1,k}(\alpha))_{i,j} := (-(\mathbf{Q}^{(k+1)})_{i,i} + \alpha)\mathbb{E}_{(k+1,i)}\left[\int_0^{\tau_{D^c_{k+1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (k,j))dt\right]$$

where for each $k \geq 0$,

$$D_k = \bigcup_{n=k}^{\infty} L_n.$$

Our first lemma shows how to numerically calculate each $\mathbf{R}$-matrix.

Lemma 2.1 *The matrices $\mathbf{R}_{k+1,k}(\alpha)$, for $k \geq 0$, satisfy the following recursion: for each integer $k \geq 1$,*

$$\mathbf{R}_{k+1,k}(\alpha) = \mathcal{M}^{(k+1)}[\alpha\mathbf{I}^{(k)} - \mathbf{Q}^{(k)} - \mathbf{R}_{k,k-1}(\alpha)\Lambda^{(k-1)}]^{-1}$$

where $\mathbf{R}_{1,0}(\alpha) = \mathcal{M}^{(0)}(\alpha\mathbf{I}^{(0)} - \mathbf{Q}^{(0)})^{-1}$.

Proof The argument follows with reasoning similar to that described on pages 270 through 272 of [8]: defining, for each $n \geq 1$, and each $m \in \{0, 1, 2, \ldots, n-1\}$, the matrix $\mathbf{R}_{n,m}(\alpha)$ as

$$(\mathbf{R}_{n,m}(\alpha))_{i,j} := (-(\mathbf{Q}^{(n)})_{i,i} + \alpha)\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{D_n^c}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right]$$

it is not difficult to show, via the random-product technique as is done in [8], that

$$\mathbf{R}_{n,m}(\alpha) = \coprod_{k=n}^{m+1} \mathbf{R}_{k,k-1}(\alpha) := \mathbf{R}_{n,n-1}(\alpha)\mathbf{R}_{n-1,n-2}(\alpha)\cdots\mathbf{R}_{m+1,m}(\alpha).$$

Readers should note our usage of the coproduct symbol $\coprod$: given a collection of matrices $\{H_k\}_{k\geq 0}$, we define

$$\prod_{k=m}^{n} H_k := H_m H_{m+1} \cdots H_n$$

for $m \leq n$, while we define

$$\coprod_{k=m}^{n} H_k := H_m H_{m-1} \cdots H_n$$

for $m \geq n$.

The next step is to establish that $(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})^{-1}$ exists, for each integer $n \geq 1$ and each $\alpha \in \mathbb{C}_+$. Fix an integer $n \geq 1$, and consider an alternative CTMC $\{Y_n(t); t \geq 0\}$ whose state space is given by

$$S_n := \bigcup_{k=0}^{n+1} L_k^{(n)}$$

where $L_k^{(n)} = L_k$ for each integer $k \in \{0, 1, 2, \ldots, n\}$, and $L_{n+1}^{(n)} = \{\Delta\}$, an absorbing state. The transition rate matrix $\mathbf{Q}_n$ is similar to the transition matrix of $\{Y(t); t \geq 0\}$, except that the row corresponding to level $L_{n+1}^{(n)}$ is the zero row, and the rows corresponding to level $L_n^{(n)}$ can be expressed in block partitioned form as

$$[\mathbf{0}_{d_n \times d_0} \quad \mathbf{0}_{d_n \times d_1} \quad \ldots \quad \mathbf{0}_{d_n \times d_{n-2}} \quad \mathcal{M}^{(n)} \quad \mathbf{Q}^{(n)} \quad \Lambda^{(n)}\mathbf{e}_{d_n \times 1}]$$

where $\mathbf{e}_{m\times 1}$ is a column vector with m rows and each element equal to one, and $\mathbf{0}_{m\times n}$ is a zero matrix with m rows and n columns. We also use the notation $\mathbf{e}_{m\times 1}^{(i)}$ to represent the ith *basis vector* in $\mathbb{R}^{m\times 1}$, where the ith component of $\mathbf{e}_{m\times 1}^{(i)}$ is equal to one and all of its other components are equal to zero. Similarly, we let $e_{1\times n}^{(i)}$ denote the ith basis vector in $\mathbb{R}^{1\times n}$, which is defined in a completely analogous manner.

We can establish the invertibility of $(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})$ through working with the Laplace transforms of the transition functions of $\{Y_n(t); t \geq 0\}$. Define, for each $0 \leq m_0, m_1 \leq n$, the matrix

$$\Pi_{m_0,m_1}^{(n)}(\alpha) := [\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha)]_{1\leq i\leq d_{m_0}, 1\leq j\leq d_{m_1}}$$

where

$$\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha) := \int_0^\infty e^{-\alpha t}\mathbb{P}_{(m_0,i)}(Y_n(t) = (m_1,j))dt$$

with $\mathbb{P}_{(m_0,i)}$ denoting a conditional probability measure, given $Y_n(0) = (m_0, i)$. Fix $m \in \{0, 1, 2, \ldots, n-1\}$: applying Corollary 1.1 where $T = L_m$ yields

$$\Pi_{n,m}^{(n)}(\alpha) = \Pi_{n,n}^{(n)}(\alpha)\mathbf{R}_{n,m}(\alpha) = \Pi_{n,n}^{(n)}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\mathbf{R}_{n-1,n-2}(\alpha)\cdots\mathbf{R}_{m+1,m}(\alpha).$$

Having this fact in mind, if we now write out the Kolmogorov Forward equations associated with $\{Y_n(t); t \geq 0\}$ in terms of Laplace transforms, we see in particular that

$$\begin{aligned}\alpha\Pi^{(n)}_{n,n}(\alpha) - \mathbf{I}^{(n)} &= \Pi^{(n)}_{n,n-1}(\alpha)\Lambda^{(n-1)} + \Pi^{(n)}_{n,n}(\alpha)\mathbf{Q}^{(n)} \\ &= \Pi^{(n)}_{n,n}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)} + \Pi^{(n)}_{n,n}(\alpha)\mathbf{Q}^{(n)}\end{aligned}$$

which yields

$$\Pi^{(n)}_{n,n}(\alpha)(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)}) = \mathbf{I}^{(n)}$$

proving that the matrix $(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})$ is invertible. Finally, one can use the random-product technique as is done in [7] to show that

$$\alpha\mathbf{R}_{k+1,k}(\alpha) = \mathcal{M}^{(k+1)} + \mathbf{R}_{k+1,k}(\alpha)\mathbf{Q}^{(k)} + \mathbf{R}_{k+1,k}(\alpha)\mathbf{R}_{k,k-1}(\alpha)\Lambda^{(k-1)}$$

meaning we can express $\mathbf{R}_{k+1,k}(\alpha)$ in terms of $\mathbf{R}_{k,k-1}(\alpha)$, thus proving the result. □

We are now ready to proceed with the main results of this section. Further associated with $\{Y(t); t \geq 0\}$ is a stochastic process $\{\overline{X}(t); t \geq 0\}$, where for each real $t \geq 0$,

$$\overline{X}(t) := \sup_{0 \leq s \leq t} X(s)$$

which represents the *maximum level* achieved by $\{Y(t); t \geq 0\}$ over the interval $[0, t]$: in [11], the authors refer to $\{\overline{X}(t); t \geq 0\}$ as the *running maximum* process. We can further combine $\overline{X}(t)$ and $Y(t)$ by defining the stochastic process $Z(t) := (\overline{X}(t), X(t), J(t))$, which is clearly also a CTMC, whose state space $\overline{S}$ is

$$\overline{S} = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{n} L_{n,m}$$

where for each integer $n \geq 0$, and each integer $m \in \{0, 1, 2, \ldots, n\}$,

$$L_{n,m} := \{([n, m], 1), ([n, m], 2), \ldots, ([n, m], d_m - 1), ([n, m], d_m)\}.$$

Observe that state $([n, m], k)$ has level $[n, m]$ and phase k, where $k \in \{1, 2, \ldots, d_m\}$.

In Theorem 2.1 we study the marginal distributions of $\{Z(t); t \geq 0\}$ by applying Corollary 1.1 in various ways. Throughout both this section and the next, we let $\boldsymbol{\pi}_{[n,m]}(\alpha)$ denote a row vector in $\mathbb{C}^{1\times d_m}$ which is of the form

$$\boldsymbol{\pi}_{[n,m]}(\alpha) = [\pi_{([m_0,m_0],i_0),([n,m],1)}(\alpha), \quad \pi_{([m_0,m_0],i_0),([n,m],2)}(\alpha), \quad \cdots \quad \pi_{([m_0,m_0],d_{m_0}),([n,m],d_m)}].$$

Readers should note that the row vector $\boldsymbol{\pi}_{[n,m]}(\alpha)$ depends on $Z(0) = ([m_0, m_0], i_0)$, but we chose to leave this out of the notation in the interest of making the results easier to read. Observe too that we will also occasionally let $\mathbb{P}_{([m_0,m_0],i_0)}$ denote a conditional probability measure, conditioned on $Z(0) = ([m_0, m_0], i_0)$. It will always be clear from the context what is being conditioned on when we write $\mathbb{P}_x$, so we will use this notation throughout the rest of the paper without further comment.

Theorem 2.1 *Suppose $Z(0) = (m_0, m_0, i_0)$. Then*

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}}[\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)} - \mathbf{R}_{m_0,m_0-1}(\alpha)\Lambda^{(m_0-1)}]^{-1}. \tag{4}$$

Furthermore, for each $n \geq m_0 + 1$,

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[m_0,m_0]}(\alpha) \prod_{\ell=m_0}^{n-1} \Lambda^{(\ell)}[\alpha\mathbf{I}^{(\ell+1)} - \mathbf{Q}^{(\ell+1)} - \mathbf{R}_{\ell+1,\ell}(\alpha)\Lambda^{(\ell)}]^{-1}. \tag{5}$$

Finally, for each $n \geq m_0$, and each $m \in \{0, 1, 2, \ldots, n-1\}$,

$$\boldsymbol{\pi}_{[n,m]}(\alpha) = \boldsymbol{\pi}_{[n,n]}(\alpha) \prod_{\ell=n}^{m+1} \mathbf{R}_{\ell,\ell-1}(\alpha) \tag{6}$$

Proof We first prove (6). Applying (2) to $\{Z(t); t \geq 0\}$ while choosing

$$T = \bigcup_{k=0}^{n-1} L_{n,k}$$

yields, for each state $([n,m],j) \in T$,

$$\boldsymbol{\pi}_{([n,m],j)}(\alpha) = \sum_{i=1}^{d_n} \pi_{([n,n],i)}(\alpha)(q([n,n],i)+\alpha)\mathbb{E}_{([n,n],i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,m],j))dt\right]. \quad (7)$$

Next, observe that for each $i \in \{1,2,\ldots,d_n\}$,

$$\begin{aligned}
&(-(\mathbf{Q}^{(n)})_{i,i}+\alpha)\mathbb{E}_{([n,n],i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,m],j))dt\right] \\
&= (-(\mathbf{Q}^{(n)})_{i,i}+\alpha)\mathbb{E}_{(n,i)}\left[\int_0^{\tau_{T^c}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right] \\
&= (\mathbf{R}_{n,m}(\alpha))_{i,j} \qquad (8)
\end{aligned}$$

and applying what was learned in (8) to (7) yields, upon further simplification,

$$\boldsymbol{\pi}_{[n,m]}(\alpha) = \boldsymbol{\pi}_{[n,n]}(\alpha)\mathbf{R}_{n,m}(\alpha) = \boldsymbol{\pi}_{[n,n]}(\alpha)\prod_{\ell=n}^{m+1}\mathbf{R}_{\ell,\ell-1}(\alpha) \quad (9)$$

proving (6).

The next step is to establish (5). Applying again (2) to $\{Z(t); t \geq 0\}$ while choosing $T = L_{n,n}$ yields, upon simplifying,

$$\begin{aligned}
\boldsymbol{\pi}_{[n,n]}(\alpha) &= \boldsymbol{\pi}_{[n-1,n-1]}\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1} + \boldsymbol{\pi}_{[n,n-1]}(\alpha)\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1} \\
&= \boldsymbol{\pi}_{[n-1,n-1]}\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1} + \boldsymbol{\pi}_{[n,n]}(\alpha)\mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)})^{-1}
\end{aligned}$$

meaning

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[n-1,n-1]}(\alpha)\Lambda^{(n-1)}(\alpha\mathbf{I}^{(n)} - \mathbf{Q}^{(n)} - \mathbf{R}_{n,n-1}(\alpha)\Lambda^{(n-1)})^{-1}$$

and by repeatedly iterating this equality, we establish (5).

It remains to derive (4). Thinking now of $\{Z(t); t \geq 0\}$ as being governed by a countable collection of independent, homogeneous Poisson processes as we described in Section 1, we observe that for each phase $k \in \{1,2,\ldots,d_{m_0}\}$, we have that for each $t > 0$,

$$\begin{aligned}
&\mathbf{1}(Z(t) = ([m_0,m_0],k)) \\
&= \mathbf{1}(Z(t) = ([m_0,m_0],k), \tau_{L^c_{m_0,m_0}} > t) \qquad (10) \\
&+ \sum_{j=1}^{d_{m_0-1}}\sum_{\ell=1}^{d_{m_0}}\int_0^t \mathbf{1}(Z(s-) = ([m_0,m_0-1],j), \tau_{L^c_{m_0,m_0}}(s) > t, Z(t) = ([m_0,m_0],k))N_{([m_0,m_0-1],j),([m_0,m_0],\ell)}(ds)
\end{aligned}$$

Taking expectations of both sides of (10), while further applying the Campbell-Mecke formula to the right-hand-side gives

$$\begin{aligned}
&\mathbb{P}_{([m_0,m_0],i_0)}(Z(t) = ([m_0,m_0],k)) \\
&= \mathbb{P}_{([m_0,m_0],i_0)}(Z(t) = ([m_0,m_0],k), \tau^c_{L_{m_0,m_0}} > t) \qquad (11) \\
&+ \sum_{j=1}^{d_{m_0-1}}\sum_{\ell=1}^{d_{m_0}}\int_0^t \mathbb{P}_{([m_0,m_0],j)}(Z(s) = ([m_0,m_0-1],j))\mathbb{P}_{([m_0,m_0],\ell)}(\tau_{L^c_{m_0,m_0}} > t-s, Z(t-s) = ([m_0,m_0],k))(\Lambda^{(m_0-1)})_{j,\ell}ds
\end{aligned}$$

and after multiplying both sides of (11) by $e^{-\alpha t}$ and integrating with respect to t over $[0,\infty)$, we get

$$\pi_{[m_0,m_0],k}(\alpha) = \mathbb{E}_{([m_0,m_0],i_0)}\left[\int_0^{\tau_{L^c_{m_0,m_0}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([m_0,m_0],k))ds\right]$$

$$+\sum_{j=1}^{d_{m_0-1}}\sum_{\ell=1}^{d_{m_0}}\pi_{([m_0,m_0-1],j)}(\alpha)(\Lambda^{(m_0-1)})_{j,\ell}\mathbb{E}_{([m_0,m_0],\ell)}\left[\int_0^{\tau^c_{L_{m_0,m_0}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([m_0,m_0],k))dt\right]$$

which can be stated in matrix form as

$$\begin{aligned}\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) &= \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)})^{-1} + \boldsymbol{\pi}_{[m_0,m_0-1]}(\alpha)\Lambda^{(m_0-1)}(\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)})^{-1}\\ &= \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)})^{-1} + \boldsymbol{\pi}_{[m_0,m_0]}(\alpha)\mathbf{R}_{m_0,m_0-1}(\alpha)\Lambda^{(m_0-1)}(\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)})^{-1}.\end{aligned}$$

Finally, solving for $\boldsymbol{\pi}_{[m_0,m_0]}(\alpha)$ yields

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}}(\alpha\mathbf{I}^{(m_0)} - \mathbf{Q}^{(m_0)} - \mathbf{R}_{m_0,m_0-1}(\alpha)\Lambda^{(m_0-1)})^{-1}$$

which proves (4), and completes the proof of Theorem 2.1. □

3 Markov Processes of M/G/1 Type

We close by studying the joint distribution of the running minimum level, the level, and the phase of a level-dependent Markov Process of M/G/1-type at a fixed time t. Suppose now that $\{Y(t); t \geq 0\}$ represents a level-dependent Markov process of M/G/1-type whose state space S can be expressed in terms of a countable union of levels:

$$S = \bigcup_{n=0}^{\infty} L_n$$

where, for each integer $n \geq 0$,

$$L_n := \{(n,1),(n,2),\ldots,(n,d_n-1),(n,d_n)\},$$

where each d_n is a positive integer that varies with n. Just as before, we express $Y(t)$ as $((X(t), J(t))$, where $X(t)$ and $J(t)$ denotes the current level and phase of the process at time t, respectively. We express the transition rate matrix $\mathbf{Q}$ of $\{Y(t); t \geq 0\}$ in block-partitioned form as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_{0,0} & \mathbf{A}_{0,1} & \mathbf{A}_{0,2} & \mathbf{A}_{0,3} & \mathbf{A}_{0,4} & \cdots \\ \mathbf{A}_{1,0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{A}_{1,3} & \mathbf{A}_{1,4} & \cdots \\ \mathbf{0}_{d_2\times d_0} & \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{A}_{2,4} & \cdots \\ \mathbf{0}_{d_3\times d_0} & \mathbf{0}_{d_3\times d_1} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Observe that for each integer $i \geq 0$ and each $j \geq i-1$, $\mathbf{A}_{i,j} \in \mathbb{R}^{d_i\times d_j}$ contains the transition rates corresponding to transitions from states in L_i to states in L_j. Again we assume that for each state $x \in S$, there exists two states $y, z \in S$ (that may depend on x) such that $q(x,y) > 0$ and $q(z,x) > 0$.

Just as in Section 2, there is an important family of '$\mathbf{R}$-matrices' $\{\mathbf{R}_{\ell,m}(\alpha)\}_{m\geq 1, 0\leq \ell<m}$ such that for each integer $m \geq 1$ and each integer $\ell \in \{0,1,\ldots,m-1\}$

$$(\mathbf{R}_{\ell,m}(\alpha))_{i,j}(\alpha) := (-(\mathbf{A}_{\ell,\ell})_{i,i} + \alpha)\mathbb{E}_{(\ell,i)}\left[\int_0^{\tau_{C_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right]$$

where for each integer $m \geq 1$,

$$C_m = \bigcup_{n=0}^{m} L_n.$$

Our analysis of Markov processes of M/G/1-type also involves a close study of a family of '**G**-matrices' $\{\mathbf{G}_{n,m}(\alpha)\}_{0 \leq n < m}$ where for each integer $n \geq 1$ and each integer $m \in \{0, 1, \ldots, n-1\}$,

$$(\mathbf{G}_{n,m}(\alpha))_{i,j} = \mathbb{E}_{(n,i)}\left[\mathbf{1}(Y(\tau_{L_m}) = (m,j))e^{-\alpha \tau_{L_m}}\right].$$

Our next lemma, Lemma 3.1, shows how to express all **R**-matrices in terms of **G**-matrices.

Lemma 3.1 *For each integer $m \geq 1$, and each integer $\ell \in \{0, 1, 2, \ldots, m-1\}$, we have*

$$\mathbf{R}_{\ell,m}(\alpha) = \sum_{k=m}^{\infty} \mathbf{A}_{\ell,k}\mathbf{G}_{k,m}(\alpha)\left[\alpha \mathbf{I}^{(m)} - \sum_{k=m}^{\infty} \mathbf{A}_{m,k}\mathbf{G}_{k,m}(\alpha)\right]^{-1} \tag{12}$$

where we follow the convention that $\mathbf{G}_{m,m}(\alpha) := \mathbf{I}^{(m)}$. Furthermore, for each $m \geq 0$, and each $k > m$,

$$\mathbf{G}_{k,m}(\alpha) = \prod_{\ell=k}^{m+1} \mathbf{G}_{\ell,\ell-1}(\alpha) := \mathbf{G}_{k,k-1}(\alpha)\mathbf{G}_{k-1,k-2}(\alpha)\cdots\mathbf{G}_{m+1,m}(\alpha) \tag{13}$$

*and the family of **G**-matrices $\{\mathbf{G}_{k+1,k}(\alpha)\}$ satisfy the following recursive scheme: for each integer $k \geq 1$,*

$$\mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k,k-1}\left[\alpha \mathbf{I}^{(k)} - \mathbf{A}_{k,k} - \sum_{i=k+1}^{\infty} \mathbf{A}_{k,i} \prod_{j=i}^{k+1} \mathbf{G}_{j,j-1}(\alpha)\right]^{-1} \tag{14}$$

Proof We follow the line of reasoning given in the unpublished manuscript [8]. First, we define the collection of matrices $\{\mathbf{N}_m(\alpha)\}_{m \geq 1}$, where for each integer $m \geq 1$, and each integer $i, j \in \{1, 2, \ldots, d_m\}$ (where possibly $i = j$),

$$(\mathbf{N}_m(\alpha))_{i,j} := \mathbb{E}_{(m,i)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right].$$

Applying a first-step analysis argument shows that

$$\begin{aligned}(\mathbf{N}_m(\alpha))_{i,j} &= \frac{\mathbf{1}(i=j)}{q((m,i)) + \alpha} + \sum_{k \neq i} \frac{q((m,i),(m,k))}{q((m,i)) + \alpha}(\mathbf{N}_m(\alpha))_{k,j} \\ &+ \sum_{k=m+1}^{\infty}\sum_{n=1}^{d_k} \frac{q((m,i),(k,n))}{q((m,i)) + \alpha}\mathbb{E}_{(k,n)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right].\end{aligned} \tag{15}$$

We can use the strong Markov property at the stopping time τ_{L_m} to further simplify the remaining expectations found in (15): indeed,

$$\begin{aligned}&\mathbb{E}_{(k,n)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right] \\ &= \sum_{\ell=1}^{d_m} \mathbb{E}_{(k,n)}[\mathbf{1}(Y(\tau_{L_m}) = (m,\ell))e^{-\alpha \tau_{L_m}}]\mathbb{E}_{(m,\ell)}\left[\int_0^{\tau_{L_{m-1}}} e^{-\alpha t}\mathbf{1}(Y(t) = (m,j))dt\right] \\ &= \sum_{\ell=1}^{d_m} (\mathbf{G}_{k,m}(\alpha))_{n,\ell}(\mathbf{N}_m(\alpha))_{\ell,j}.\end{aligned} \tag{16}$$

Plugging (16) into (15), then expressing (15) (while remembering that $\mathbf{G}_{m,m}(\alpha) = \mathbf{I}^{(m)}$) we get

$$\alpha \mathbf{N}_m(\alpha) = \mathbf{I}^{(m)} + \sum_{k=m}^{\infty} \mathbf{A}_{m,k} \mathbf{G}_{k,m}(\alpha) \mathbf{N}_m(\alpha), \tag{17}$$

which implies

$$\left[\alpha \mathbf{I}^{(m)} - \sum_{k=m}^{\infty} \mathbf{A}_{m,k} \mathbf{G}_{k,m}(\alpha) \right] \mathbf{N}_m(\alpha) = \mathbf{I}^{(m)}$$

meaning

$$\mathbf{N}_m(\alpha) = \left[\alpha \mathbf{I}^{(m)} - \sum_{k=m}^{\infty} \mathbf{A}_{m,k} \mathbf{G}_{k,m}(\alpha) \right]^{-1}. \tag{18}$$

We are now ready to derive (12). From the definition of $\mathbf{R}_{\ell,m}(\alpha)$, we can see from applying both first-step analysis and the strong Markov property that

$$\mathbf{R}_{\ell,m}(\alpha) = \sum_{k=m}^{\infty} \mathbf{A}_{\ell,k} \mathbf{G}_{k,m}(\alpha) \mathbf{N}_m(\alpha). \tag{19}$$

Plugging (18) into (19) yields (12).

The next step is to establish (13). Fix an integer $m \geq 0$ and an integer $k > m$. Using again the strong Markov property, we get

$$\mathbf{G}_{k,m}(\alpha) = \mathbf{G}_{k,k-1}(\alpha) \mathbf{G}_{k-1,m}(\alpha),$$

and by a simple induction argument, we get

$$\mathbf{G}_{k,m}(\alpha) = \prod_{\ell=k}^{m+1} \mathbf{G}_{\ell,\ell-1}(\alpha)$$

which establishes (13).

It remains to derive (14). Fix $i \in \{1, 2, \ldots, d_k\}$ and $j \in \{1, 2, \ldots, d_{k-1}\}$,

$$(\mathbf{G}_{k,k-1}(\alpha))_{i,j} = \frac{q((k,i),(k-1,j))}{q((k,i)) + \alpha} + \sum_{\ell \neq i} \frac{q((k,i),(k,\ell))}{q((k,i)) + \alpha} (\mathbf{G}_{k,k-1}(\alpha))_{\ell,j}$$
$$+ \sum_{m=k+1}^{\infty} \sum_{\ell=1}^{d_m} \frac{q((k,i),(m,\ell))}{q((k,i)) + \alpha} (\mathbf{G}_{m,k-1}(\alpha))_{\ell,j}$$

or, in matrix form,

$$\alpha \mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k,k-1} + \sum_{m=k}^{\infty} \mathbf{A}_{k,m} \mathbf{G}_{m,k-1}(\alpha). \tag{20}$$

Applying (13) to (20) shows that

$$\alpha \mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k,k} + \sum_{i=k+1}^{\infty} \mathbf{A}_{k,i} \left(\prod_{j=i}^{k+1} \mathbf{G}_{j,j-1}(\alpha) \right) \mathbf{G}_{k,k-1}(\alpha) \tag{21}$$

and solving for $\mathbf{G}_{k,k-1}(\alpha)$ in (21) gives

$$\mathbf{G}_{k,k-1}(\alpha) = \mathbf{A}_{k+1,k} \left[\alpha \mathbf{I}^{(k+1)} - \mathbf{A}_{k+1,k+1} - \sum_{i=k+1}^{\infty} \mathbf{A}_{k+1,i} \prod_{j=i}^{k+1} \mathbf{G}_{j,j-1}(\alpha) \right]^{-1}$$

which proves (14). □

While Lemma 3.1 is theoretically interesting, it is only practically useful if the $\mathbf{G}$-matrices can be calculated numerically. It is not clear in general if there is a way to calculate these matrices, but they can be calculated if we impose additional assumptions on $\{Y(t); t \geq 0\}$. Suppose, for instance, that there exists an integer $n_0 \geq 1$ large enough such that $\mathbf{A}_{n,k} = \mathbf{A}_{k-n}$ for all $n \geq n_0$ and $k \geq n-1$. Under this additional assumption, one can see that $\mathbf{G}_{n,n-1}(\alpha) = \mathbf{G}(\alpha)$ for each $n \geq n_0$, where

$$\mathbf{G}(\alpha) := \mathbf{G}_{n_0,n_0-1}(\alpha).$$

As explained in [8], the matrix $\mathbf{G}(\alpha)$ is the pointwise limit of a sequence of matrices $\{\mathbf{G}(N,\alpha)\}_{N\geq 0}$, where $\mathbf{G}(0,\alpha) = \mathbf{0}_{d_{n_0}\times d_{n_0}}$, and for each integer $N \geq 0$,

$$\mathbf{G}(N+1,\alpha) = (\alpha\mathbf{I}^{(d_{n_0})} - \mathbf{A}_0)^{-1}\left[\mathbf{A}_{-1} + \sum_{n=1}^{\infty}\mathbf{A}_n\mathbf{G}(N,\alpha)^n\right].$$

We are now ready to set up and establish the main result of this section. We associate with $\{Y(t); t \geq 0\}$ the stochastic process $\{\underline{X}(t); t \geq 0\}$ where for each $t \geq 0$,

$$\underline{X}(t) := \inf_{0\leq s\leq t} X(s)$$

which represents the *running minimum level* achieved by $\{Y(t); t \geq 0\}$ over the interval $[0,t]$. Next, for each $t \geq 0$ we define $Z(t) := (\underline{X}(t), X(t), J(t))$, and just as was the case in the previous section, $\{Z(t); t \geq 0\}$ is a CTMC with state space

$$\underline{S} = \bigcup_{n=0}^{\infty}\bigcup_{m=n}^{\infty} L_{n,m}$$

where for each integer $n \geq 0$ and each integer $m \geq n$,

$$L_{n,m} := \{([n,m],1),([n,m],2),\ldots,([n,m],d_m-1),([n,m],d_m)\}.$$

In our next result, Theorem 3.1, we show how to derive the Laplace transforms of the transition functions associated with $\{Z(t); t \geq 0\}$.

Theorem 3.1 *Suppose* $Z(0) = (m_0, m_0, i_0)$*. Then*

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}_{1\times d_{m_0}}^{(i_0)}(\alpha\mathbf{I}^{(m_0)} - \mathbf{A}_{m_0,m_0} - \mathbf{R}_{m_0,m_0+1}(\alpha)\mathbf{A}_{m_0+1,m_0})^{-1}. \tag{22}$$

Furthermore, for each integer $n \in \{0,1,\ldots,m_0-1\}$,

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[m_0,m_0]}(\alpha)\prod_{\ell=m_0-1}^{n}\mathbf{A}_{\ell+1,\ell}\left[\alpha\mathbf{I}^{(\ell)} - \mathbf{A}_{\ell,\ell} - \mathbf{R}_{\ell,\ell+1}(\alpha)\mathbf{A}_{\ell+1,\ell}\right]^{-1}. \tag{23}$$

Finally, for each integer $n \in \{0,1,\ldots,m_0-1,m_0\}$ *and each integer* $m \geq n$,

$$\boldsymbol{\pi}_{[n,m+1]}(\alpha) = \sum_{k=n}^{m}\boldsymbol{\pi}_{[n,k]}(\alpha)\mathbf{R}_{k,m+1}(\alpha). \tag{24}$$

Proof We will first establish invertibility of $(\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n})$ for any integer $n \in \{0,1,\ldots,m_0-1,m_0\}$ and each $\alpha \in \mathbb{C}_+$. To do so, we use a strategy similar to that used in Lemma 2.1. Fix an integer $n \geq 1$ and consider an alternative CTMC $\{Y_n(t); t \geq 0\}$ whose state space is given by

$$S_n := \bigcup_{k=n-1}^{\infty} L_k^{(n)}$$

where $L_{n-1}^{(n)} = \{\Delta\}$ (a single absorbing state) and for each $k \geq n$, $L_k^{(n)} := L_k$. The transition rate matrix $\mathbf{Q}_n$ of $\{Y_n(t); t \geq 0\}$ is such that the row corresponding to level $L_{n-1}^{(n)}$ is a row containing all zeros, the rows corresponding to $L_n^{(n)}$ can be expressed in block-partitioned form as

$$[\mathbf{A}_{n,n-1}\mathbf{e} \quad \mathbf{A}_{n,n} \quad \mathbf{A}_{n,n+1} \quad \mathbf{A}_{n,n+2} \quad \cdots]$$

and for each $k \geq n+1$, the rows corresponding to level $L_k^{(n)}$ the same as the rows corresponding to L_k in the transition rate matrix of $\{Y(t); t \geq 0\}$.

Next, we define, for each m_0 and $m_1 \geq n$ the matrix

$$\Pi_{m_0,m_1}^{(n)}(\alpha) := [\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha)]_{1 \leq i \leq d_{m_0}, 1 \leq j \leq d_{m_1}}$$

where

$$\pi_{(m_0,i),(m_1,j)}^{(n)}(\alpha) := \int_0^\infty e^{-\alpha t} \mathbb{P}_{(m_0,i)}(Y_n(t) = (m_1, j)) dt.$$

Observe that by applying Corollary 1.1 while choosing

$$T = \bigcup_{k=n+1}^{\infty} L_k$$

yields

$$\Pi_{n,n+1}^{(n)}(\alpha) = \Pi_{n,n}^{(n)}(\alpha) \mathbf{R}_{n,n+1}(\alpha).$$

With this in mind, after writing out the Kolmogorov Forward equations associated with $\{Y_n(t); t \geq 0\}$ in terms of Laplace transforms, we see that

$$\begin{aligned}
\Pi_{n,n}^{(n)}(\alpha)(\alpha \mathbf{I}^{(n)} - \mathbf{A}_{n,n}) - \Pi_{n,n+1}(\alpha)\mathbf{A}_{n+1,n} &= \mathbf{I}^{(n)} \\
\Pi_{n,n}^{(n)}(\alpha)(\alpha \mathbf{I}^{(n)} - \mathbf{A}_{n,n}) - \Pi_{n,n}(\alpha)\mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n} &= \mathbf{I}^{(n)}
\end{aligned}$$

which yields

$$\Pi_{n,n}^{(n)}(\alpha)(\alpha \mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n}) = \mathbf{I}^{(n)}$$

proving that the matrix $(\alpha \mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n})$ is invertible. A similar argument can be made for the case where $n = 0$, but in that case we can establish invertibility with the forward equations of $\{Y(t); t \geq 0\}$.

We now prove (24): fix $n \in \{0, 1, \ldots, m_0 - 1, m_0\}$ and suppose $m \geq n$. Applying Corollary 1.1 to $\{Z(t); t \geq 0\}$ while choosing

$$T = \bigcup_{\ell=m+1}^{\infty} L_{n,\ell}$$

yields, for each $j \in \{1, 2, \ldots, d_{m+1}\}$,

$$\pi_{([n,m+1],j)}(\alpha) = \sum_{\ell=n}^{m} \sum_{i=1}^{d_\ell} \pi_{([n,\ell],i)}(\alpha)(q([n,\ell],i) + \alpha)\mathbb{E}_{([n,\ell],i)} \left[\int_0^{\tau_{T^c}} e^{-\alpha t} \mathbf{1}(Z(t) = ([n, m+1], j)) dt \right]. \tag{25}$$

Furthermore, for each $\ell \in \{n, n+1, \ldots, m\}$ and each $i \in \{1, 2, \ldots, d_\ell\}$,

$$\begin{aligned}
&(q([n,\ell],i) + \alpha)\mathbb{E}_{([n,\ell],i)} \left[\int_0^{\tau_{T^c}} e^{-\alpha t} \mathbf{1}(Z(t) = ([n, m+1], j)) dt \right] \\
&= (-(\mathbf{A}_{\ell,\ell})_{i,i} + \alpha)\mathbb{E}_{(\ell,i)} \left[\int_0^{\tau_{C_m}} e^{-\alpha t} \mathbf{1}(Y(t) = (m+1, j)) dt \right] \\
&= (\mathbf{R}_{\ell,m}(\alpha))_{i,j}.
\end{aligned} \tag{26}$$

Applying (26) to (25), then writing (25) in matrix form yields

$$\boldsymbol{\pi}_{[n,m+1]}(\alpha) = \sum_{\ell=n}^{m} \boldsymbol{\pi}_{[n,\ell]}(\alpha)\mathbf{R}_{\ell,m+1}(\alpha)$$

which proves (24).

We next establish (22). for each $j \in \{1, 2, \ldots, d_{m_0}\}$,

$$\begin{aligned}
&\mathbf{1}(Z(t) = (m_0, m_0, j)) \\
&= \mathbf{1}(Z(t) = (m_0, m_0, j), \tau_{L^c_{m_0,m_0}} > t) \qquad (27) \\
&+ \sum_{k=1}^{d_{m_0}} \sum_{\ell=1}^{d_{m_0}} \int_0^t \mathbf{1}(Z(s-) = (m_0, m_0+1, k), \tau_{L^c_{m_0,m_0}}(s) > t, Z(t) = (m_0, m_0, j)) N_{(m_0,m_0+1,k),(m_0,m_0,\ell)}(ds)
\end{aligned}$$

Just as we did in the proof of Theorem 2.1, after taking the expectation of both sides of (27), applying the Campbell-Mecke formula to the right-hand-side, multiplying by $e^{-\alpha t}$, integrating, and then simplifying, we get

$$\boldsymbol{\pi}_{[m_0,m_0]}(\alpha) = \mathbf{e}^{(i_0)}_{1\times d_{m_0}} (\alpha \mathbf{I}^{(m_0)} - \mathbf{A}_{m_0,m_0} - \mathbf{R}_{m_0,m_0+1}(\alpha)\mathbf{A}_{m_0+1,m_0})^{-1}$$

proving (22).

It remains to derive (23). Fix an integer $n \in \{0, 1, \ldots, m_0-1\}$: applying Corollary 1.1 to $\{Z(t); t \geq 0\}$ while choosing $T = L_{n,n}$ reveals that

$$\begin{aligned}
\pi_{[n,n],j}(\alpha) &= \sum_{i=1}^{d_{n+1}} \pi_{[n+1,n+1],i}(\alpha)(q([n+1,n+1],i)+\alpha)\mathbb{E}_{([n+1,n+1],i)}\left[\int_0^{\tau_{L^c_{n,n}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,n],j))dt\right] \\
&+ \sum_{i=1}^{d_{n+1}} \pi_{[n,n+1],i}(\alpha)(q([n,n+1],i)+\alpha)\mathbb{E}_{([n,n+1],i)}\left[\int_0^{\tau_{L^c_{n,n}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,n],j))dt\right].
\end{aligned} \qquad (28)$$

Again, for $k \in \{n, n+1\}$, $i \in \{1, 2, \ldots, d_k\}$, and $j \in \{1, 2, \ldots, d_n\}$,

$$\begin{aligned}
&(q([k,n+1],i)+\alpha)\mathbb{E}_{([k,n+1],i)}\left[\int_0^{\tau_{L^c_{n,n}}} e^{-\alpha t}\mathbf{1}(Z(t) = ([n,n],j))dt\right] \\
&= (-(\mathbf{A}_{n+1,n+1})_{i,i} + \alpha)\mathbb{E}_{(n+1,i)}\left[\int_0^{\tau_{L^c_{n}}} e^{-\alpha t}\mathbf{1}(Y(t) = (n,j))dt\right].
\end{aligned} \qquad (29)$$

Plugging (29) into (28) and simplifying further shows that

$$\boldsymbol{\pi}_{[n,n](\alpha)} = \boldsymbol{\pi}_{[n+1,n+1]}(\alpha)\mathbf{A}_{n+1,n}[\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n}]^{-1} + \boldsymbol{\pi}_{[n,n+1]}(\alpha)\mathbf{A}_{n+1,n}[\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n}]^{-1}$$

and by writing $\boldsymbol{\pi}_{[n,n+1]}(\alpha)$ in terms of $\boldsymbol{\pi}_{[n,n]}(\alpha)$ and solving for $\boldsymbol{\pi}_{[n,n]}(\alpha)$ yields

$$\boldsymbol{\pi}_{[n,n]}(\alpha) = \boldsymbol{\pi}_{[n+1,n+1]}(\alpha)\mathbf{A}_{n+1,n}\left[\alpha\mathbf{I}^{(n)} - \mathbf{A}_{n,n} - \mathbf{R}_{n,n+1}(\alpha)\mathbf{A}_{n+1,n}\right]^{-1}.$$

which yields, upon repeated iterations of this equality, (23), thus proving Theorem 3.1. □

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