Tightened Miller-Tucker-Zemlin Inequalities for the Target Visitation and Related Problems

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Abstract

Miller-Tucker-Zemlin (MTZ) inequalities are known to eliminate subtours in the traveling salesman problem (TSP). Given a collection of n objects (cities) that must be each visited exactly once by a salesman, the appeal of these inequalities is that they are few in number, being of order $O(n^2)$. A drawback is that they give weak continuous relaxations. Efforts to improve their strength have yielded modest success. This paper uses conditional logic to obtain strengthened MTZ-type inequalities for different problems that seek to eliminate subtours. Our efforts initially focus on the Target Visitation Problem (TVP), which is a generalization of both the Linear Ordering Problem (LOP) and the TSP, where we are able to exploit variable structure to obtain strengthenings. We then apply the approach to other problems, including the LOP, a capacitated vehicle routing problem, the TSP, and the quadratic TSP. Computational experience is provided to demonstrate the merits on the TVP.

Keywords: target visitation, traveling salesman, linear ordering.

1 Introduction

This paper uses conditional logic to tighten the Miller-Tucker-Zemlin (MTZ) inequalities of [17] that are used to eliminate subtours in the traveling salesman problem (TSP). Our study is motivated by a generalization of the TSP, known as the Target Visitation Problem (TVP) (see [10, 11, 12]), which differs from the TSP in that, in addition to incurring the usual inter-city travel costs, also realizes rewards as in the Linear Ordering Problem (LOP). The structure of the TVP necessitates additional variables beyond the TSP, and these variables are the source of our tightenings.

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The TVP can be thought of as a hybrid between the TSP and LOP. Given a collection of n objects, all three problems desire an optimal permutation, with the difference being the manner in which the objectives are defined. For each pair of distinct objects i and j, the TSP incurs a cost, say c_{ij}, for having object i *immediately* precede object j whereas the LOP incurs a reward, say r_{ij}, for having object i *anywhere* precede object j. For the TSP, unlike the LOP, the last object in a permutation is considered to immediately precede the first object. The TVP incurs both the costs and rewards, and arises in such instances as unmanned aerial vehicles tasked for surveillance, attack, search and rescue, and the delivery of supplies in both military and civilian applications ([10]). We are concerned with the general, asymmetric TSP and TVP for which c_{ij} need not necessarily equal c_{ji} for the $(i, j), i < j$, pairs.

The TSP and LOP are known to be NP-hard, so that the more general TVP is also NP-hard. The challenge for each problem is the exponential number of permutations that must be considered. A distinguishing feature of the TVP is that, since the objective is a function of both the *immediate* and *anywhere* relative ordering of each pair of distinct objects i and j, the mathematical representation must accommodate both ordering traits. Specifically, for each such pair of objects, the TSP typically defines a binary variable, say x_{ij}, to represent whether object i *immediately* precedes object j, while the LOP defines a binary variable, say y_{ij}, to represent whether object i *anywhere* precedes object j. In order to record both the costs and rewards, the TVP uses both sets of variables. The TVP generalizes both the TSP and LOP in the sense that it reduces to the first when the objective coefficients on the variables y_{ij} are all 0, and it reduces to the second when the objective coefficients on the variables x_{ij} are all 0.

As we discuss in Section 2, various formulations of the TSP are available in the literature that include both sets of variables x_{ij} and y_{ij}. The idea behind these forms is to use the variables y_{ij} to tighten the continuous relaxations of the associated binary optimization problems. Our analysis, while motivated by the TVP, is then directly applicable to such forms of the TSP, yielding theoretically tighter cuts.

The TSP, LOP, and TVP can be commonly envisioned on a graph having n nodes, with each node corresponding to an object. The TSP and TVP seek optimal Hamiltonian circuits while the LOP seeks an optimal Hamiltonian path. All three problems share the task of eliminating subtours of nodes. Here, a subtour is defined as a Hamiltonian circuit formed on a proper subset of two or more of the nodes. For the much-studied TSP, strategies other than MTZ inequalities have been used to accomplish this task, with the work of [1] being a classical example. The "subtour elimination constraints" of [1], known as the Dantzig-Fulkerson-Johnson inequalities, provide tight representations, but are exponential in number. In contrast, the MTZ inequalities, as well as subsequent tightenings due to [2], are far fewer in number, being of order $O(n^2)$, but are known to afford weak relaxations. For the LOP, "3-dicycle" inequalities have been used by [8, 9] and [16], and the number of these inequalities is of order $O(n^3)$. The 3-dicycle inequalities, and tightenings thereof [18], have also been used [21] with mixed success for the TSP. These inequalities yield considerably tighter relaxations than the MTZ inequalities, but the relaxations are more expensive to solve. The challenge for all three problems is to develop formulations that have tight continuous relaxations which enable strong bounds that are not prohibitively expensive to compute. The key is to balance the tradeoff between the strength of the bounds and the computational effort required to obtain them.

We exploit the two sets of variables x_{ij} and y_{ij} found within the TVP to tighten the MTZ inequalities, and we perform computational tests to evaluate their merits. Our approach consists of two steps. The first step uses the conditional logic of [19] embedded within the reformulation-linearization technique (RLT) to strategically compute quadratic inequalities that are valid for the TVP. Each such inequality is obtained by multiplying a binary expression with a linear restriction that is not necessarily valid for the TVP, but is "conditionally" valid when the binary expression realizes the value 1. The second step follows the work of [15] to suitably surrogate these quadratic inequalities to obtain linear inequalities. It turns out that these new inequalities reduce the solution times for the TVP as compared to alternate applications of the MTZ inequalities. From a theoretical perspective, while [6] showed that the MTZ inequalities and tightenings by [2] can be described in terms of aggregates of a special family of "3-dicycle" inequalities, we show that our cuts cannot be explained in this manner. As a result, our cuts are able to garner additional strength beyond these other inequalities.

Our approach, while simple in form, allows us to intuitively explain MTZ inequality tightenings for the TSP, generate new MTZ-type inequalities for the LOP, motivate MTZ inequalities for a capacitated vehicle routing problem (CVRP), improve lifted-inequalities for the TSP, and derive tightened MTZ inequalities for the quadratic traveling salesman problem (QTSP). Specifically, we show that a partial (weakened) application of the conditional logic arguments that does not use the variables y_{ij} yields an intuitive derivation of a tightening of [2] for the TSP, and an application that does not use the variables x_{ij} yields a new and most concise known formulation for the LOP. The logic also allows for a simple derivation of strengthened inequalities for a single-depot CVRP. The lifted inequalities for the TSP and the strengthened inequalities for the QTSP exploit variable products of the forms $x_{ik}y_{kj}$ and $x_{ik}x_{kj}$, respectively.

This paper is organized as follows. In the next section, we provide a brief literature review of formulations for each of the LOP, TSP, and TVP. This review serves as the basis for motivating our improved MTZ inequalities. Section 3 then derives our strengthened cuts for the TVP, explains the work of [2] for the TSP, motivates the new formulation for the LOP, and demonstrates the applicability to the CVRP. This section also shows our earlier-mentioned result that, unlike the standard MTZ inequalities and the strengthenings by [2], our cuts cannot be explained in terms of aggregates of a family of 3-dicycle inequalities. In Section 4, we use different implementations of our cuts to construct new representations of the TVP, and provide computational experience for various sets of objective function coefficients. Of significance here is that our cuts tighten the continuous relaxations and expedite the solution process; this result is intriguing as the standard MTZ inequalities are known to be weak for the TSP. Section 5 closes with conclusions and avenues for future research, as well as our lifted inequalities for the TSP and strengthened inequalities for the QTSP.

2 Literature Review

In this section, we review known formulations for each of the LOP, TSP, and TVP in order to establish a knowledge base for demonstrating the merits of Section 3. In fact, we obtain in Section 3 improved formulations for all of these problems. Throughout the remainder of the paper, we let n denote the number of objects and assume that all indices run from 1 to n unless otherwise stated.

2.1 Linear Ordering Problem

As described earlier, given a collection of n objects, the LOP seeks a permutation (ordering) that yields the maximum reward. Rewards are computed in terms of pairs of distinct objects i and j, with a reward r_{ij} realized when object i anywhere precedes object j in the permutation. The LOP can be formulated [8, 9, 16] as below.

$$\text{LOP1: maximize} \sum_i \sum_{j \neq i} r_{ij} y_{ij}$$

subject to

$$y_{ij} + y_{jk} + y_{ki} \leq 2 \qquad \forall\ (i,j,k) \text{ distinct} \tag{1}$$

$$y_{ij} + y_{ji} = 1 \qquad \forall\ (i,j),\ i < j \tag{2}$$

$$y_{ij} \text{ binary} \qquad \forall\ (i,j),\ i \neq j \tag{3}$$

The decision variables are the $2\binom{n}{2} = n(n-1)$ binary y_{ij} defined so that

$$y_{ij} = \begin{cases} 1 & \text{if object } i \textit{ anywhere} \text{ precedes object } j \text{ in the permutation} \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ (i,j),\ i \neq j.$$

Problem LOP1 is explained as follows. Inequalities (1) and equations (2) combine to enforce a valid permutation (eliminate subtours), with equations (2) stating that for each (i,j), either object i precedes object j or object j precedes object i. Inequalities (1) are the earlier-mentioned "3-dicycle" inequalities that are of order $O(n^3)$ in number. Then the objective function records the total reward for each of the n-factorial possible permutations. Of course, we have the option to remove half the variables found within LOP1 by substituting $y_{ji} = 1 - y_{ij}$ for all $(i,j),\ i < j$, throughout the objective function and inequalities (1), and to then remove equations (2).

2.2 Traveling Salesman Problem

As explained earlier, given n objects, the TSP seeks a permutation of the objects that minimizes an overall cost. For each pair of distinct objects i and j, a cost c_{ij} is incurred for having object i immediately precede object j in the permutation. Unlike the LOP, for each permutation, the last object is defined to immediately precede the first so that an associated cost is incurred. The TSP has been extensively studied, and a variety of mathematical representations are found in the literature. A classical formulation of particular interest to this study is the following, due to [17].

$$\text{TSP1: minimize} \sum_i \sum_{j \neq i} c_{ij} x_{ij}$$

subject to

$$\sum_{j \neq i} x_{ij} = 1 \qquad \forall\, i \tag{4}$$

$$\sum_{i \neq j} x_{ij} = 1 \qquad \forall\, j \tag{5}$$

$$u_j - u_i \geq (2-n) + (n-1)x_{ij} \qquad \forall\, (i,j),\ i,j \geq 2,\ i \neq j \tag{6}$$

$$x_{ij} \text{ binary} \qquad \forall\, (i,j),\ i \neq j \tag{7}$$

The decision variables are the $n(n-1)$ binary x_{ij} defined so that

$$x_{ij} = \begin{cases} 1 & \text{if object } i \textit{ immediately} \text{ precedes object } j \text{ in the permutation} \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, (i,j),\ i \neq j,$$

together with $(n-1)$ "dummy variables" $u_j, j = 1, \ldots, n-1$, defined (as explained below) to eliminate subtours. Equations (4) enforce that each object i has a single object following it, and equations (5) enforce that each object j has a single object preceding it. Inequalities (6) are the MTZ inequalities that serve to eliminate subtours. There exist $2\binom{n-1}{2} = (n-1)(n-2)$ such MTZ inequalities. The objective function records the cost of each possible permutation.

The TSP draws its name from the scenario of a salesman who must conduct a "tour" of n cities; the tour consists of the salesman beginning at some home city, visiting each city exactly once, and then returning to the home city. With this interpretation, each city takes the role of an object, and the cost of traveling from city i to city j is c_{ij}. The binary variables then have that $x_{ij} = 1$ if the salesman travels from city i to city j, and 0 otherwise. The salesman's tour comprises a Hamiltonian circuit of the cities.

There is a notable difference between the LOP and TSP. As earlier mentioned, when envisioned on a graph having n nodes, the LOP seeks a Hamiltonian path while the TSP seeks a Hamiltonian circuit. The reason for this difference is that, given any permutation, the TSP has a nonzero objective coefficient associated with having the last object immediately precede the first, while the LOP does not. The effect is that the TSP is concerned with relative locations of adjacent objects, so that every permutation has the same objective value as that of the $(n-1)$ other permutations whose objects have the same adjacencies. As a result, it is usual to assume, without loss of generality, that object 1 is located first in the permutation (i.e., object 1 is the home city). This assumption cannot be made with the LOP. Consequently, the LOP must consider n-factorial permutations while the TSP need only consider $(n-1)$-factorial.

The variables u_j in (6) have an interesting interpretation. Consistent with the above explanation that city 1 is the first city in the tour, for each $j \in \{2, \ldots, n\}$, the variable u_j can be thought of as representing the location of city j in the permutation, with $u_j = k$ indicating that city j is located in the k^{th} position after the first ($u_j = k$ indicates that city j is located in position $(k+1)$). Then we have

$$1 \leq u_j \leq n-1 \ \ \forall\, j \geq 2, \tag{8}$$

and these inequalities can be enforced as explicit restrictions.

The paper of [2] strengthens inequalities (6) and (8) in the sense that the continuous relaxation of TSP1 obtained by relaxing (7) to $x_{ij} \geq 0$ for all $(i,j),\ i \neq j$, will be tightened. For the instances of the TSP in which $n \geq 4$, this paper strengthens (6) to

$$u_j - u_i \geq (2-n) + (n-1)x_{ij} + (n-3)x_{ji} \ \ \forall\ (i,j),\ i,j \geq 2,\ i \neq j, \tag{9}$$

which is inequality (6) with the right side increased by the nonnegative quantity $(n-3)x_{ji}$. Inequalities (9) are valid by the following logic: if $x_{ji} = 0$, then (9) reduces to (6) while if $x_{ji} = 1$, then $x_{ij} = 0$ and $u_j - u_i = -1$. The paper of [2] strengthens (8) to

$$2 - x_{1j} + (n-3)x_{j1} \leq u_j \leq (n-2) + (3-n)x_{1j} + x_{j1} \ \ \forall\ j \geq 2, \tag{10}$$

which is inequality (8) with the left side increased by the nonnegative quantity $(1-x_{1j})+(n-3)x_{j1}$, and with the right side decreased by the nonnegative quantity $(n-3)x_{1j} + (1-x_{j1})$. For each $j \geq 2$, the two inequalities of (10) are valid by the following logic. There exist three possible feasible realizations of x_{1j} and x_{j1} : $x_{1j} = x_{j1} = 0$, $x_{1j} = (1-x_{j1}) = 0$, and $(1-x_{1j}) = x_{j1} = 0$. For these three possibilities, (10) gives us that $2 \leq u_j \leq n-2$, $u_j = (n-1)$, and $u_j = 1$, respectively, verifying the two inequalities. Therefore, we can replace (6) of TSP1 with (9) and (10) to obtain a tightened valid representation of the TSP.

Notably, the variables y_{ij} found within the LOP have also been used within the TSP ([6, 7, 18, 21]) to eliminate subtours. Different methods relate the variables x_{ij} and y_{ij}, but all enforce that

$$x_{ij} \leq y_{ij} \ \ \forall\ (i,j),\ i,j \geq 2,\ i \neq j. \tag{11}$$

In the more recent works of [18] and [21], the MTZ inequalities (6) of Problem TSP1 are replaced with those restrictions of (1) and (2) that have $i,j,k \geq 2$, together with (11), as given in the formulation below.

TSP2: minimize $\sum_i \sum_{j \neq i} c_{ij}x_{ij}$

subject to

(4), (5), (7), (11)

$$y_{ij} + y_{jk} + y_{ki} \leq 2 \qquad \forall\ (i,j,k) \text{ distinct},\ i,j,k \geq 2 \tag{12}$$

$$y_{ij} + y_{ji} = 1 \qquad \forall\ (i,j),\ 2 \leq i < j \tag{13}$$

In formulating TSP2, for each $j \in \{2, \ldots, n\}$, the variable y_{1j} can be envisioned as having been fixed to the value 1 and the value y_{j1} to the value 0 within (1) and (2) to reflect that city 1 is located first in the permutation. This fixing of variables allows for the simplification of these two families of restrictions to (12) and (13), respectively. In addition, the paper of [18] explains that restrictions (3) are not needed, and that the elimination of subtours amongst cities 2 through n, as enforced by the constraints of TSP2, implies that no subtour can exist which includes city 1. (Note that (7), (11), and (13) combine to enforce that $0 \leq y_{ij} \leq 1\ \forall\ (i,j),\ i,j \geq 2,\ i \neq j$, so that these bounding restrictions are not included in TSP2, as they do not tighten the linear programming relaxation.)

Computational experience of [18] showed that, while Problem TSP2 affords a much tighter lower bound on the optimal objective function value to the TSP than does TSP1, the additional $O(n^3)$ inequalities of (12), together with the additional variables y_{ij}, make the problem more expensive to solve. However, TSP2 performed well when the variables y_{ij} were already present in the formulation to model precedence constraints dictating that certain objects must be located prior to others in the permutation.

The paper of [21] derived a tightening of the inequalities (12) by increasing, for each (i, j, k), the left side by the nonnegative quantity x_{ji} to obtain

$$y_{ij} + y_{jk} + y_{ki} + x_{ji} \leq 2 \;\; \forall \, (i, j, k) \text{ distinct}, \;\; i, j, k \geq 2. \tag{14}$$

(As pointed out in [21], for each such (i, j, k), either of the variables x_{kj} or x_{ik} could have been used in lieu of x_{ji} within (14) to yield the same family of inequalities by symmetry.) The formulation TSP3 results.

$$\text{TSP3: minimize} \left\{ \sum_{i} \sum_{j \neq i} c_{ij} x_{ij} : (4), (5), (7), (11), (13), (14) \right\}$$

We conclude this section by mentioning that the paper of [6] related the variables u_j of the MTZ inequalities (6), (8), (9), (10) to the variables y_{ij} by

$$u_j = \sum_{\substack{i \geq 2 \\ i \neq j}} y_{ij} + 1 \;\; \forall \, j \geq 2. \tag{15}$$

2.3 Target Visitation Problem

As discussed earlier, the TVP is a hybrid of the LOP and TSP that incurs both a reward when an object i anywhere precedes a second object j, and a cost when it immediately precedes object j. As a result, formulations for the TVP can mimic TSP2 and TSP3 [10, 11, 12] by suitably modifying the objective function to obtain TVP0 and TVP1 below, respectively.

$$\text{TVP0: maximize} \left\{ \sum_{i \geq 2} \sum_{\substack{j \geq 2 \\ j \neq i}} r_{ij} y_{ij} - \sum_{i} \sum_{j \neq i} c_{ij} x_{ij} : (4), (5), (7), (11) - (13) \right\}$$

$$\text{TVP1: maximize} \left\{ \sum_{i \geq 2} \sum_{\substack{j \geq 2 \\ j \neq i}} r_{ij} y_{ij} - \sum_{i} \sum_{j \neq i} c_{ij} x_{ij} : (4), (5), (7), (11), (13), (14) \right\}$$

For both TVP0 and TVP1, the variables y_{ij} and x_{ij} retain the same definitions as in TSP1 and TSP2, as do the rewards r_{ij} and costs c_{ij}. The costs c_{ij} are negated in the objective function since the problem is a maximization. For the sake of consistency and simplicity, we carry through to the TVP the assumption from the TSP that object 1 is located first in the permutation. Since this assumption cannot be made without loss of generality, we use when necessary, that object 1 has been predefined as a "dummy" with all associated objective coefficients corresponding to y_{1j} and y_{j1} set to 0 so that $r_{1j} = r_{j1} = 0 \;\; \forall \, j \geq 2$. Then we do not include these $2(n-1)$ variables within either TVP0 or TVP1.

Before progressing to our strengthened MTZ inequalities in the next section, we note here that alternate formulations are available for the TVP, some transferable from the TSP as in [6, 7], and others specifically designed for the TVP as in [11]. These formulations employ triple-subscripted variables, so that the variable number is of order $O(n^3)$. Examples include defining, for each (i, j, k) distinct, a binary w_{ijk} : to denote having object i *anywhere* precede object j and object j *anywhere* precede object k, to denote having object i *immediately* precede object j and object j *anywhere* precede object k, to denote having object i *anywhere* precede object j and object j *immediately* precede object k, and to denote having object i *immediately* precede object j and object j *immediately* precede object k. Another example is to have, for each (i, j, k) with i and j distinct and with $k \in \{1, \ldots, n-1\}$, a binary w_{ijk} to denote that object i precedes object j by exactly k locations in the permutation. In contrast, we adopt an alternate approach of maintaining a smaller problem size by restricting the number of variables to be of order $O(n^2)$.

3 Strengthened MTZ Inequalities via Conditional Logic

We now present the conditional logic for computing strengthened MTZ inequalities. While motivated by the TVP, the approach will not only allow us to obtain additional inequalities for the TVP, but also for the TSP, LOP, and a single-depot capacitated vehicle routing problem (CVRP). Included here are the inequalities (9) and (10) of [2] for the TSP, a variation of (6) that uses y_{ij} instead of x_{ij}, and tightened MTZ inequalities for a CVRP.

For the TVP, we exploit the three sets of variables x_{ij}, y_{ij}, and u_j found within Problem TVP1. We first use the conditional logic of [19] to compute quadratic inequalities that are valid for the TVP. These restrictions are obtained by multiplying binary expressions of the variables x_{ij} and y_{ij} by linear restrictions in the variables u_j. The restrictions in u_j are only "conditionally" valid for the TVP in the sense that their correctness is conditioned on the multiplying binary expressions. The restrictions are strategically computed so that we can use the surrogation idea of [15] to obtain linear inequalities.

The section is divided into three subsections. The first subsection gives our approach for computing strengthened MTZ inequalities for the TVP. The second subsection shows that, unlike known MTZ forms, our inequalities have the distinguishing theoretical property that they cannot be explained in terms of aggregates of (14). The third subsection demonstrates the general applicability by deriving (9) and (10), and the inequalities (6) expressed in terms of y_{ij}. We also intuitively derive some known, rather-sophisticated MTZ-type inequalities for the CVRP, as well as suggest new inequalities.

3.1 Strengthened MTZ Inequalities for the TVP

To begin, recall the interpretation of the variables u_j which states that $u_j = k$ indicates that object j is located in the k^{th} position after the first within the permutation. Then, for each

$(i,j),\ i,j \geq 2,\ i \neq j$, we can make the following conditional statements.

$$\begin{array}{ll} \text{If } x_{ij} = 1 & \text{then } u_j - u_i = 1. \\ \text{If } x_{ji} = 1 & \text{then } u_j - u_i = -1. \\ \text{If } y_{ij} - x_{ij} = 1 & \text{then } u_j - u_i \geq 2. \\ \text{If } y_{ji} - x_{ji} = 1 & \text{then } u_j - u_i \geq 2 - n. \end{array}$$

Each of the expressions x_{ij}, x_{ji}, $y_{ij} - x_{ij}$, and $y_{ji} - x_{ji}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the TVP. Consequently, we can multiply each such expression by its associated consequence to obtain the following four quadratic restrictions that are valid for the TVP for every $(i,j),\ i,j \geq 2,\ i \neq j$.

$$x_{ij}(u_j - u_i = 1), x_{ji}(u_j - u_i = -1), (y_{ij} - x_{ij})(u_j - u_i \geq 2), (y_{ji} - x_{ji})(u_j - u_i \geq 2 - n)$$

The validity holds true for the following reason. For each such product, if the first expression is 0, then the product trivially holds. Otherwise, the first expression is 1 and the product holds by the associated conditional statement.

The surrogation process to obtain a linear inequality consists of simply adding the four quadratic restrictions together. The following inequalities result upon using (13) to let $y_{ji} = 1 - y_{ij}$.

$$u_j - u_i \geq (2 - n) + ny_{ij} - x_{ij} + (n-3)x_{ji} \ \ \forall\, (i,j),\ i,j \geq 2,\ i \neq j \tag{16}$$

These inequalities are our desired tightened version of (9). For each $(i,j),\ i,j \geq 2,\ i \neq j$, the inequality of (16) can be obtained by adding the nonnegative expression $n(y_{ij} - x_{ij})$ to the right side of the corresponding inequality in (9). Inequalities (16) recognize that if object i *anywhere* precedes object j, but does not *immediately* precede it, then object j is located at least two positions after object i.

Remark. *The conditional logic statements are consistent with our earlier assumption that some object (object 1) is located first in the permutation. Otherwise, if we had not fixed the first position and accordingly instead let $u_j = 0$ indicate that object j is located in position 1 then, for given (i,j), the first statement will fail if object j is in position 1, and the second statement will fail if object j is in the last position n. The last two statements will also fail since the antecedents are no longer necessarily binary, as* (11) *no longer holds true when object j is in position 1.*

3.2 Comparison with Known MTZ Inequalities

There is an intriguing distinction between the MTZ inequalities (6) and (9), and the new inequalities (16) that we computed using conditional logic. In the spirit of [6], the MTZ inequalities (6) and (9) can be viewed as an aggregation of a (weakened) version of the 3-dicycle inequalities (14) in the presence of (11), (13), and (15); the inequalities (16) cannot be viewed in this manner. To explain, suppose for some $(i,j),\ i,j \geq 2,\ i \neq j$, that we have the following family of valid inequalities for the TVP:

$$\alpha_{ij} + y_{jk} + y_{ki} + \beta_{ji} \leq 2 \ \ \forall\, k \geq 2,\ k \neq i,j, \tag{17}$$

which are equivalent by $y_{jk} + y_{kj} = 1$ for all $k \neq i, j$ of (2) to

$$\alpha_{ij} - y_{kj} + y_{ki} + \beta_{ji} \leq 1 \ \ \forall\, k \geq 2,\ k \neq i, j. \tag{18}$$

Here, α_{ij} and β_{ji} represent "dummy" variables. Then we can sum over the $(n-3)$ inequalities of (18) to obtain

$$(n-3)\alpha_{ij} - (\sum_{\substack{k\geq 2\\ k\neq j}} y_{kj} - y_{ij}) + (\sum_{\substack{k\geq 2\\ k\neq i}} y_{ki} - (1 - y_{ij})) + (n-3)\beta_{ji} \leq (n-3),$$

where we have used (13) to set $y_{ji} = 1 - y_{ij}$. This inequality simplifies by (15) to

$$u_j - u_i \geq (2-n) + (n-3)\alpha_{ij} + 2y_{ij} + (n-3)\beta_{ji}. \tag{19}$$

Now, if $\alpha_{ij} = x_{ij}$ and $\beta_{ji} = 0$, then inequalities (17) are a weakened version of the 3-dicycle inequalities found in (14) so that (19) is valid, and adding the inequality $0 \geq 2(x_{ij} - y_{ij})$ of (11) to (19) yields (6). Similarly, if $\alpha_{ij} = x_{ij}$ and $\beta_{ji} = x_{ji}$, then inequalities (17) are a weakened version of inequalities found in (14) so that (19) is again valid, and adding the inequality $0 \geq 2(x_{ij} - y_{ij})$ of (11) gives (9). Finally, suppose that we further tighten (17) to have $\alpha_{ij} = y_{ij}$ and $\beta_{ji} = x_{ji}$ as in (14). Then the valid inequality (19) becomes

$$u_j - u_i \geq (2-n) + (n-1)y_{ij} + (n-3)x_{ji}. \tag{20}$$

which is tighter than either of the two above-mentioned versions of (19). However, this inequality is weaker than (16) because adding the inequality $0 \geq (x_{ij} - y_{ij})$ of (11) to (16) gives (20).

3.3 Alternate MTZ Inequality Derivations

It is insightful to observe that the conditional logic approach affords an intuitive derivation of the strengthened MTZ inequalities (9) and (10) of [2] for the TSP in the absence of the variables y_{ij}, a modified version of the MTZ inequalities (6) that uses y_{ij} instead of x_{ij}, and strengthened MTZ inequalities of [2] and [13] for a singe-depot capacitated vehicle routing problem (CVRP). As we will see, the second result will give rise to the most concise known formulation of the LOP.

1. Relative to the TSP, first consider (9). For each $(i, j),\ i, j \geq 2,\ i \neq j$, we have the following conditional statements.

   $$\begin{array}{ll} \text{If } x_{ij} = 1 & \text{then } u_j - u_i = 1. \\ \text{If } x_{ji} = 1 & \text{then } u_j - u_i = -1. \\ \text{If } 1 - x_{ij} - x_{ji} = 1 & \text{then } u_j - u_i \geq 2 - n. \end{array}$$

   Each of the expressions x_{ij}, x_{ji}, and $1 - x_{ij} - x_{ji}$ found in the antecedents of these statements is binary for all feasible solutions to the TSP. Then multiplying each such expression by its associated consequence and summing the resulting three quadratic expressions gives us (9).

Next consider (10). For each $j \geq 2$, we have the following conditional statements.

$$\begin{array}{ll} \text{If } x_{1j} = 1 & \text{then } u_j = 1. \\ \text{If } x_{j1} = 1 & \text{then } u_j = n-1. \\ \text{If } 1 - x_{1j} - x_{j1} = 1 & \text{then } 2 \leq u_j \leq n-2. \end{array}$$

Each of the expressions x_{1j}, x_{j1}, and $1 - x_{1j} - x_{j1}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the TSP. Then, upon separately considering the two inequalities in the third statement, we can multiply each such expression by its associated consequence and sum to obtain (10). Here, $2 \leq u_j$ and then $u_j \leq n-2$ of the third statement give rise to the left and right inequalities of (10), respectively.

2. Relative to expressing (6) in terms of y_{ij} instead of x_{ij}, suppose that, as in the LOP, we do not restrict object 1 to be located first in the permutation. Then we can extend the definition of u_j to have $u_1 = k$ indicate that object 1 is located in the k^{th} position after the first, and to have $u_j = 0$ indicate that object j is located first in the permutation. This extended definition changes (8) to

$$0 \leq u_j \leq n-1 \;\; \forall \, j \geq 1.$$

As a result, for each (i,j), $i \neq j$, we have the following conditional statements.

$$\begin{array}{ll} \text{If } y_{ij} = 1 & \text{then } u_j - u_i \geq 1. \\ \text{If } 1 - y_{ij} = 1 & \text{then } u_j - u_i \geq 1-n. \end{array}$$

Each of the expressions y_{ij} and $1 - y_{ij}$ found in the antecedents of these conditional statements is binary for all feasible solutions to the LOP. Then we can multiply each expression by its associated consequence and sum to obtain

$$u_j - u_i \geq (1-n) + n y_{ij} \;\; \forall \, (i,j), \; i \neq j. \tag{21}$$

Remark. *Unlike the conditional statements of Section 3.1, this second application holds true when no object is assumed to have been a priori located first in the permutation. Thus, inequalities* (21) *are valid for all* (i,j), $i \neq j$, *while inequalities* (16) *require that* $i,j \geq 2$.

The MTZ-type inequalities (21) lead to the most concise known formulation of the LOP. This formulation is obtained by replacing the $n(n-1)(n-2)$ inequalities (1) of LOP1 with the $n(n-1)$ inequalities of (21). The result is as follows.

$$\text{LOP2: maximize} \left\{ \sum_i \sum_{j \neq i} r_{ij} y_{ij} : (2), (3), (21) \right\} \tag{22}$$

Problem LOP2 has $n(n-1)$ variables y_{ij}, n variables u_j, $\frac{n(n-1)}{2}$ equalities (2), and $n(n-1)$ inequalities (21), in addition to the binary restrictions on y_{ij}. As noted earlier for Problem LOP1, we can remove half the variables y_{ij} by substituting $y_{ji} = 1 - y_{ij}$

for all $(i, j),\ i < j$, throughout the problem, and then remove equations (2). The resulting formulation will have $\frac{n(n-1)}{2}$ variables y_{ij}, n variables u_j, and the $n(n-1)$ inequalities (21), along with the binary restrictions on the remaining variables y_{ij}. The below proposition formally establishes Problem LOP2 as a valid formulation of the LOP.

Proposition 1. *Problem LOP2 is a valid formulation of the LOP in that the feasible solutions to* (22) *correspond to the n-factorial permutations of the set of n objects.*

Proof. It is sufficient to show that the constraints of Problem LOP2 imply (1). Toward this end, consider any distinct (i, j, k), and sum the three inequalities of (21) given by

$$u_j - u_i \geq (1-n) + ny_{ij},\ u_k - u_j \geq (1-n) + ny_{jk},\ u_i - u_k \geq (1-n) + ny_{ki}$$

to obtain $0 \geq 3 + n(y_{ij} + y_{jk} + y_{ki} - 3)$. Then (3) gives us that $y_{ij} + y_{jk} + y_{ki} \leq 2$ must hold true. □

3. The single-depot CVRP is a generalization of the TSP that seeks to route m vehicles amongst the n cities so that each city is visited exactly once, so that each route begins and ends at home city (depot) 1 with no subtours, and so that all city demands for a product are met. Here, each of the m vehicles has the same capacity Q for the product and each city j has some demand $q_j > 0$. The objective is to minimize the overall cost while meeting the demands of all cities. There are different formulations for this CVRP (see [22] for example), with a form attributable to [14] closely related to TSP1. This form uses the same binary variables x_{ij} and continuous variables u_j and the same costs c_{ij} as TSP1, but changes the right side of (4) for $i = 1$ and the right side of (5) for $j = 1$ from 1 to m to reflect that m vehicles are leaving and entering city 1, respectively. Of particular importance to this study are modified versions of (6) and (8) to accommodate CVRP. Inequalities of [13], corrected from work of [14], change (6) to
$$u_j - u_i \geq (q_j - Q) + Qx_{ij}\ \ \forall\ (i, j),\ i, j \geq 2,\ i \neq j. \tag{23}$$
In addition, inequalities (8) change to
$$q_j \leq u_j \leq Q\ \ \forall\ j \geq 2. \tag{24}$$
Inequalities (23) and (24) are found in [22]. In this context, we can think of each variable u_j for $j \geq 2$ as the amount of vehicle capacity that has been collectively allocated to the cities on the tour from city 1 through city j. Observe that if each city has demand $q_j = 1$ and there exists a single vehicle $m = 1$ having capacity $Q = n - 1$, then the CVRP simplifies to the TSP, and (23) and (24) reduce to (6) and (8), respectively.

A result of [2], corrected and clarified by [13], presents strengthened versions of (23) and (24). These strengthened inequalities are
$$u_j - u_i \geq (q_j - Q) + Qx_{ij} + (Q - q_i - q_j)x_{ji}\ \ \forall\ (i, j),\ i, j \geq 2,\ i \neq j, \tag{25}$$

and

$$q_j + \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ij} \leq u_j \leq Q - (Q - \max_{\substack{i\geq 2\\ i\neq j}}\{q_i\} - q_j)x_{1j} - \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ji} \;\; \forall\, j \geq 2, \qquad (26)$$

respectively. The work of [13] uses coefficient lifting on x_{j1} to obtain (25) from (23), maintains the left inequalities of (26) as given by [2], and establishes the right inequalities of (26) by first showing the validity of $u_j \leq Q - \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ji}$ for each j, and then using coefficient lifting on x_{1j}. Notably, these inequalities can be instead readily derived using conditional logic. For (25), we have the following conditional statements associated with each $(i, j),\ i, j \geq 2,\ i \neq j$.

If $x_{ij} = 1$	then $u_j - u_i = q_j$.
If $x_{ji} = 1$	then $u_j - u_i = -q_i$.
If $1 - x_{ij} - x_{ji} = 1$	then $u_j - u_i \geq q_j - Q$.

Each of the expressions x_{ij}, x_{ji}, and $1 - x_{ij} - x_{ji}$ found in the antecedents of these statements is binary for all feasible solutions to the CVRP. Then multiplying each such expression by its associated consequence, and summing, gives us (25). Next consider (26). For each $j \geq 2$, we have the following conditional statements.

If $x_{1j} = 1$	then $u_j = q_j$.
If $(1 - x_{1j})x_{j1} = 1$	then $q_j + \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ij} \leq u_j \leq Q$.
If $(1 - x_{1j})(1 - x_{j1}) = 1$	then $q_j + \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ij} \leq u_j \leq Q - \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ji}$.

(Observe that, unlike the TSP, we included quadratic expressions in two of the above antecedents because it is possible to have a vehicle visit only a single city j so that $x_{1j} = x_{j1} = 1$.) Each of the expressions x_{1j}, $x_{j1}(1 - x_{1j})$, and $(1 - x_{1j})(1 - x_{j1})$ found in the antecedents of these conditional statements is binary for all feasible solutions to the CVRP. Then, upon separately considering the two inequalities in the second and third statements, we can multiply each such expression by its associated consequence and sum to obtain

$$q_j + \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ij} \leq u_j \leq q_j x_{1j} + (x_{1j} - 1)(\sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ji} - Q) \;\; \forall\, j \geq 2, \qquad (27)$$

where we have used that $x_{1j} \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ij} = x_{j1} \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ji} = 0$ for all $j \geq 2$.

For each $j \geq 2$, the left inequalities of (26) and (27) are the same. The right inequality of (27) is tighter than that of (26) since $x_{1j} \sum_{\substack{i\geq 2\\ i\neq j}} q_i x_{ji} \leq x_{1j} \max_{\substack{i\geq 2\\ i\neq j}}\{q_i\}$, though (27) is nonlinear since it contains the quadratic expression $x_{1j} \sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji}$. However, the expression $(x_{1j} - 1) \sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji}$ within (27) is bounded above by each of $(x_{1j} \max_{\substack{i \geq 2 \\ i \neq j}} \{q_i\} - \sum_{\substack{i \geq 2 \\ i \neq j}} q_i x_{ji})$ and 0. Substituting the first upper bound into (27) gives the right inequality of (26) while substituting the second upper bound changes the right inequality of (26) to $u_j \leq Q + x_{1j}(q_j - Q)$, which is an alternate strengthened form of the right inequality of (24).

Remark. *The derivations of* (25) *and* (26) *presented above can be modified to reflect each vehicle having a limitation measured in terms of factors other than product capacity, such as travel distance or transportation cost. For these related problems, the variables u_j will again represent the amount of vehicle capacity that has been collectively allocated to the cities on the tour from city 1 through city j, and Q will again represent the vehicle capacity. To elaborate, for each $(i, j), i \neq j$, let T_{ij} denote the consumption of vehicle capacity (distance or cost) that is incurred in traveling from city i to city j. Also, for each $j \geq 2$, let v_{1j} and v_{j1} denote the minimum "consumption" path from city 1 to city j and from city j to city 1, respectively, with $v_{11} = 0$. Then we have the following conditional statements for each $(i, j),\ i, j \geq 2,\ i \neq j$.*

$$\begin{array}{ll} \text{If } x_{ij} = 1 & \text{then } u_j - u_i = T_{ij}. \\ \text{If } x_{ji} = 1 & \text{then } u_j - u_i = -T_{ji}. \\ \text{If } 1 - x_{ij} - x_{ji} = 1 & \text{then } u_j - u_i \geq v_{1j} - Q. \end{array}$$

Each of the expressions x_{ij}, x_{ji}, and $1 - x_{ij} - x_{ji}$ found in the antecedents of these statements is binary for all feasible solutions to the CVRP. Applying the multiplication and surrogation process as before, we get the inequalities

$$u_j - u_i \geq (v_{1j} - Q) + (Q - v_{1j} + T_{ij}) x_{ij} + (Q - v_{1j} - T_{ji}) x_{ji} \quad \forall\, (i, j),\ i, j \geq 2,\ i \neq j,$$

which are in the spirit of (25)*. For each $j \geq 2$, we also have the conditional statements*

$$\begin{array}{ll} \text{If } x_{1j} = 1 & \text{then } u_j = T_{1j}. \\ \text{If } (1 - x_{1j}) x_{j1} = 1 & \text{then } \sum_{\substack{i \geq 2 \\ i \neq j}} (v_{1i} + T_{ij}) x_{ij} \leq u_j \leq Q. \\ \text{If } (1 - x_{1j})(1 - x_{j1}) = 1 & \text{then } \sum_{\substack{i \geq 2 \\ i \neq j}} (v_{1i} + T_{ij}) x_{ij} \leq u_j \leq Q - \sum_{\substack{i \geq 2 \\ i \neq j}} (v_{i1} + T_{ji}) x_{ji}. \end{array}$$

As with the computation of (27)*, each of the expressions x_{1j}, $x_{j1}(1 - x_{1j})$, and $(1 - x_{1j})(1 - x_{j1})$ found in the antecedents of these conditional statements is binary for all feasible solutions to the CVRP. Upon separately considering the two inequalities in the second and third statements, we can multiply each such expression by its associated consequence and sum to obtain*

$$T_{1j} x_{1j} + \sum_{\substack{i \geq 2 \\ i \neq j}} (v_{1i} + T_{ij}) x_{ij} \leq u_j \leq T_{1j} x_{1j} + (x_{1j} - 1)\Big(\sum_{\substack{i \geq 2 \\ i \neq j}} (v_{i1} + T_{ji}) x_{ji} - Q\Big) \quad \forall\, j \geq 2, \quad (28)$$

where we have used that $x_{1j}\sum_{\substack{i\geq 2\\ i\neq j}}(v_{1i}+T_{ij})x_{ij} = x_{j1}\sum_{\substack{i\geq 2\\ i\neq j}}(v_{i1}+T_{ji})x_{ji} = 0$ *for all* $j \geq 2$.

For each $j \geq 2$, *the right inequality of* (28) *is quadratic because it contains the expression* $x_{1j}\sum_{\substack{i\geq 2\\ i\neq j}}(v_{i1}+T_{ji})x_{ji}$. *But the expression* $(x_{1j}-1)\sum_{\substack{i\geq 2\\ i\neq j}}(v_{i1}+T_{ji})x_{ji}$ *within* (28) *is bounded above by each of* $(x_{1j}\max_{\substack{i\geq 2\\ i\neq j}}\{v_{i1}+T_{ji}\} - \sum_{\substack{i\geq 2\\ i\neq j}}(v_{i1}+T_{ji})x_{ji})$ *and 0. Substituting these upper bounds into the right inequalities of* (28) *gives the left and right inequalities below, respectively.*

$$
\begin{aligned}
u_j &\leq Q - (Q - \max_{\substack{i\geq 2\\ i\neq j}}\{v_{i1}+T_{ji}\} - T_{1j})x_{1j} - \sum_{\substack{i\geq 2\\ i\neq j}}(v_{i1}+T_{ji})x_{ji} \\
\text{and } \ u_j &\leq T_{1j}x_{1j} + Q(1-x_{1j}) \qquad\qquad \forall\, j \geq 2
\end{aligned}
\tag{29}
$$

Variations of the above implementation can lead to different inequalities than those stated. For example, for each $j \geq 2$, *we can redefine the scalar* v_{1i} *for* $i \geq 2, i \neq j$, *in the consequences of the last two conditional statements to denote the minimum consumption path from city 1 to city i that does* not *involve city j, yielding cuts that dominate the left inequalities of* (28). *(If, for some* $j \geq 2$, *every path from node 1 to some node i must involve node j, then the CVRP can be simplified by setting* $x_{ij} = 0$.*) For each* $j \geq 2$, *we can also redefine the scalar* v_{i1} *for* $i \geq 2, i \neq j$, *in the consequence of the last conditional statement to denote the minimum consumption path from city i to city 1 that does* not *involve city j, recognizing that if no such path exists then we can set* $x_{ji} = 0$. *Additionally, we can decrease the consequence of the second conditional statement by* T_{j1} *to the value* $(Q - T_{j1})$. *This decrease would add the value* $-T_{j1}(1-x_{1j})x_{j1}$ *to the upper bound on* u_j *of* (28). *As this value is bounded above by* $T_{j1}(x_{1j} - x_{j1})$, *each of the two upper bounds on* u_j *found in* (29) *can be incremented by this bound. Other implementations include expanding the antecedent* $x_{1j} = 1$ *in the first statement to separately consider* $x_{1j}x_{j1} = 1$ *and* $x_{1j}(1-x_{j1}) = 1$, *and then strategically weakening consequences in order to eliminate the quadratic terms in the surrogation step.*

4 Computational Experience

In this section, we examine the computational effectiveness of using MTZ inequalities within Problem TVP1 to solve the TVP, paying particular attention to the utility of our new (16). Toward this end, we compare four different formulations. The first two forms are the base cases of TVP0 and TVP1 that use no MTZ inequalities. A comparison of these forms will provide insight into the usefulness of the tightened version of (12) given by (14). The third form, referred to as Problem TVP2, includes the known MTZ inequalities (10) and the equations (15) within TVP1. The fourth, Problem TVP3, enhances TVP2 by also including (16). Here, based on preliminary computational experience, and despite the fact that they are not needed theoretically since the MTZ inequalities eliminate subtours, we included inequalities (14) and equations (15) within each of TVP2 and TVP3. Notably, we then did not include (9) in TVP2 because the argument of Section 3.2 with $\alpha_{ij} = x_{ij}$ and $\beta_{ji} = x_{ji}$ shows these inequalities to be implied by (11) and (13) – (15). Problems TVP2 and TVP3 are stated formally below.

$$\text{TVP2: maximize} \left\{ \sum_{i\geq 2} \sum_{\substack{j\geq 2\\ j\neq i}} r_{ij} y_{ij} + \sum_{i} \sum_{j\neq i} c_{ij} x_{ij} : (4), (5), (7), (10), (11), (13) - (15) \right\}$$

$$\text{TVP3: maximize} \left\{ \sum_{i\geq 2} \sum_{\substack{j\geq 2\\ j\neq i}} r_{ij} y_{ij} + \sum_{i} \sum_{j\neq i} c_{ij} x_{ij} : (4), (5), (7), (10), (11), (13) - (16) \right\}$$

It is important to note that, while the MTZ inequalities (10) and (16) are implied by the restrictions of each of Problems TVP0 and TVP1 when the variables x_{ij} are enforced to be binary in (7), and are thus implied in TVP2 and TVP3, they are *not* implied in the continuous relaxations that are obtained by replacing these restrictions with $x_{ij} \geq 0$ for all $(i, j),\ i \neq j$. We denote the continuous relaxations to Problems TVP0, TVP1, TVP2, and TVP3 that are obtained by so relaxing (7) in each as Problems $\overline{\text{TVP0}}$, $\overline{\text{TVP1}}$, $\overline{\text{TVP2}}$, and $\overline{\text{TVP3}}$, respectively.

All of our runs were made on Clemson University's high performance computer, Palmetto, with 2 CPU cores and 20gb RAM, using the CPLEX 12.8 solver in AMPL for Linux-Intel 64 with all pre-processing options turned off. Python 2.7.6 was used to generate data files with random integer objective coefficient values obtained via a uniform distribution over the ranges specified below.

We ran a total of 440 problems of size $n = 20$, consisting of ten runs each for 11 families of problems, for each of four formulations. As the TVP reduces to the LOP when $c_{ij} = 0\ \forall\, (i, j)$, and to the TSP when $r_{ij} = 0\ \forall\, (i, j)$, our concern is with the usefulness of (10) and (14) –(16) over different coefficient ranges for c_{ij} and r_{ij}. We used 11 different interval ranges for randomly computing these coefficients. Specifically, for c_{ij} in the interval $[0, 10]$, we let r_{ij} range in each of the six intervals $[0, 10]$, $[0, 20]$,[0,50], $[0, 100]$, $[0, 200]$, and $[0, 1000]$. Similarly, for r_{ij} in the interval $[0, 10]$, we let c_{ij} range in the five intervals $[0, 20]$,[0,50], $[0, 100]$, $[0, 200]$, and $[0, 1000]$. Our output gives the averages of the ten instances within each family. Each instance was restricted to a 3600 CPU second time limit.

Table 1 records, for each of the 11 families of 10 problems, the number of instances that were optimally solved in the 3600 CPU second time limit for each of the formulations. The columns are arranged so that the first two give the ranges on the coefficients c_{ij} and r_{ij}, and the next four give the formulations used: TVP0, TVP1, TVP2, and TVP3. Each of the rows depicts a different family of coefficient ranges. The key observation of Table 1 is that Problems TVP2 and TVP3 were able to solve all instances, that TVP1 was able to solve only 9 of the 10 instances when both c_{ij} and r_{ij} were in the interval $[0, 10]$, and that TVP0 was unable to solve a total of 18 out of the 110 instances.

Table 2 gives, for each of the same 11 families of problems, the average times in seconds to solve to optimality and the average numbers of branch-and-bound nodes explored for each of the four formulations. To present meaningful results, both averages were computed only for those instances that were optimally solved for all four formulations. As with Table 1, the first two columns give the ranges on the coefficients c_{ij} and r_{ij}, and the rows list the 11 families of problems. Columns 3 through 6 give the average times to solve the integer programs TVP0, TVP1, TVP2, and TVP3, respectively, while columns 7 through 10 give the average numbers of branch-and-bound nodes for these same problems.

Five notable observations of Table 2 are the following. First, for all cases, TVP1 outperformed TVP0 in terms of both execution times and numbers of nodes explored. This superiority of TVP1 over TVP0 is consistent with the results of Table 1. Thus, the tightening of (12) to (14) is advantageous. Second, for all cases except the last one having c_{ij} in the interval $[0, 1000]$ and r_{ij} in the interval $[0, 10]$, Problem TVP2 outperformed TVP1 in both execution times and nodes explored. For this last case, TVP2 was better in terms of numbers of nodes but took a slightly longer execution time. Our conclusion here is that the MTZ inequalities (10) and the equations (15) strengthen TVP1 and lead to more efficient solvings. Third, for 8 of the 11 cases, Problem TVP3 outperformed TVP2 in both execution times and nodes explored. Thus, the new cuts (16) tend to promote improved performance. Fourth, the problems appear significantly more difficult when the c_{ij} and r_{ij} coefficients are of the same relative magnitudes. This difficulty is evident when each interval is in one of the ranges $[0, 10]$, $[0, 20]$, or $[0, 50]$, in contrast, for example, with having one range as $[0, 10]$ and the other as $[0, 1000]$. Our conjecture is that simplifications occur when one set of coefficients dominates the other because the TVP then reduces to take the form of either the TSP or LOP. Finally, and generally speaking, it appears advantageous to employ MTZ inequalities in the presence of 3-dicycle inequalities. This advantage is evidenced by TVP3 outperforming TVP1 in every case except for the last relative to both execution times and numbers of nodes. The last (simplest) case has TVP3 preferable in number of nodes but slightly worse in execution time.

We are also interested in the additional strength afforded to the continuous relaxations: by (14) in TVP1 as opposed to (12) in TVP0, by (10) and (15) as found in TVP2, and by (10), (15), and (16) as found in TVP3. Table 3 gives our results. For this table, the 11 rows again list the same 11 families of problems, and the first two columns again give the coefficient ranges for c_{ij} and r_{ij}, respectively. Column 3 gives the optimal integer values, and columns 4 through 7 give the optimal objective values to the relaxations $\overline{\text{TVP0}}$, $\overline{\text{TVP1}}$, $\overline{\text{TVP2}}$, and $\overline{\text{TVP3}}$, respectively. Column 8 gives the percentage of the gap between the continuous relaxation values of $\overline{\text{TVP1}}$ and the optimal integer values that is reduced by $\overline{\text{TVP3}}$, computed as $(\frac{\overline{\text{TVP1}}-\overline{\text{TVP3}}}{\overline{\text{TVP1}}-\text{IP}}) \times 100$, and labeled G.R. for Gap Reduction, where IP is the optimal (integer programming) objective value to the TVP. While we could have instead computed the percentage gap relative to $\overline{\text{TVP0}}$ and $\overline{\text{TVP3}}$ instead of $\overline{\text{TVP1}}$ and $\overline{\text{TVP3}}$ to show even greater improvement, our intent is to see the relative strength due to (10), (15), and (16) when compared to the tighter $\overline{\text{TVP1}}$.

The table reaffirms the theoretical result that the objective values to $\overline{\text{TVP3}}$ are the tightest, followed by $\overline{\text{TVP2}}$, $\overline{\text{TVP1}}$, and finally $\overline{\text{TVP0}}$. While the majority of this improvement occurred between $\overline{\text{TVP0}}$ and $\overline{\text{TVP1}}$, the MTZ inequalities inequalities (10) and (16), when used with (15) in $\overline{\text{TVP3}}$, also contributed strength. Interestingly, column eight shows that Problem $\overline{\text{TVP3}}$ reduced the gap between the optimal objective values to $\overline{\text{TVP1}}$ and the integer optimums between 2.75% and 31.14% on average. As evident in Table 2, these reductions had a marked affect on the overall solution times. This improvement in solution times runs counter to the generally-accepted belief that the MTZ inequalities are weak.

We believe that the success exhibited in these tables by using MTZ inequalities is due both to the relatively few numbers of $2(n-1)$, $(n-1)$, and $(n-1)(n-2)$ inequalities in (10), (15), and (16), respectively, and our not requiring additional, auxiliary variables. The paper of [20] applied a partial level-1 RLT to a formulation of the TSP using MTZ inequalities by multiplying, for each i, the equation in (4) by u_i and, for each j, the equation in (5) by u_j to generate products of the form $u_i x_{ij}$ and $u_j x_{ij}$, which were in turn linearized through the use of additional variables. While the continuous relaxations were strengthened, the overall solution times increased. In contrast, we strategically computed the quadratic inequalities so that they are readily surrogated to obtain linear restrictions, and thus eliminated the need for auxiliary variables.

Coefficient Ranges		**Number of Instances Solved**			
c_{ij}	r_{ij}	TVP0	TVP1	TVP2	TVP3
[0,10]	[0,10]	5	9	10	10
[0,10]	[0,20]	7	10	10	10
[0,10]	[0,50]	7	10	10	10
[0,10]	[0,100]	10	10	10	10
[0,10]	[0,200]	10	10	10	10
[0,10]	[0,1000]	10	10	10	10
[0,20]	[0,10]	7	10	10	10
[0,50]	[0,10]	6	10	10	10
[0,100]	[0,10]	10	10	10	10
[0,200]	[0,10]	10	10	10	10
[0,1000]	[0,10]	10	10	10	10

Table 1: Numbers of Instances Solved

5 Conclusions and Future Research

This paper used a conditional-logic approach to derive tightened MTZ-type inequalities that are used to eliminate subtours in the TSP and related problems. The approach was designed to exploit the variable structure of the TVP, but is generally applicable to various problem classes, including the LOP, TSP, and CVRP. It consists of the two steps of conditional-logic and surrogation, where the conditional-logic is strategically applied so that a surrogation is immediately available. For the TVP, the computed inequalities, unlike published restrictions, have the theoretical property that they are not explainable in terms of surrogates of the 3-dicycle inequalities (14).

We showed how various families of MTZ-type inequalities for each of the TVP, LOP, TSP, and CVRP can be intuitively derived and strengthened. Computational advantages

Coefficient Ranges		**IP Times (in Seconds)**				**Branch-and-Bound Nodes**			
c_{ij}	r_{ij}	TVP0	TVP1	TVP2	TVP3	TVP0	TVP1	TVP2	TVP3
[0,10]	[0,10]	769.41	196.56	121.21	83.93	66248.40	11017.20	3025.80	1919.00
[0,10]	[0,20]	1471.90	135.32	63.32	66.48	159489.00	8195.86	2017.86	2068.29
[0,10]	[0,50]	360.54	61.78	23.74	27.06	43860.86	4045.14	661.29	623.00
[0,10]	[0,100]	177.01	25.65	12.01	11.32	22824.50	1513.70	337.10	255.00
[0,10]	[0,200]	134.81	20.33	12.19	9.44	19481.10	1350.00	350.20	231.30
[0,10]	[0,1000]	5.63	4.29	1.71	1.30	485.00	189.90	20.10	5.40
[0,20]	[0,10]	1125.38	187.92	100.96	84.04	83289.86	8896.29	2899.00	2035.00
[0,50]	[0,10]	1386.31	252.36	123.43	108.43	95687.33	12794.00	3166.00	2711.17
[0,100]	[0,10]	119.12	33.17	20.93	17.32	6665.60	1323.90	337.40	236.70
[0,200]	[0,10]	38.26	20.13	12.87	11.20	1708.20	711.00	177.10	127.00
[0,1000]	[0,10]	2.80	2.70	3.13	3.24	50.70	36.20	14.20	14.30

Table 2: IP Times & Branch-and-Bound Nodes

Coefficient Ranges		**IP**	**Continuous Relaxation Values**				**G.R.**
c_{ij}	r_{ij}	**Values**	TVP0	TVP1	TVP2	TVP3	**%**
[0,10]	[0,10]	972.80	1009.43	1001.41	1000.89	1000.53	3.08
[0,10]	[0,20]	1940.80	1984.22	1973.19	1972.75	1972.30	2.75
[0,10]	[0,50]	4890.71	4949.06	4921.03	4920.50	4919.93	3.63
[0,10]	[0,100]	10022.40	10070.49	10055.37	10054.92	10052.81	7.76
[0,10]	[0,200]	20025.50	20075.64	20057.23	20056.80	20054.52	8.54
[0,10]	[0,1000]	100528.00	100559.23	100543.77	100543.51	100538.86	31.14
[0,20]	[0,10]	934.70	979.36	968.31	967.14	966.63	5.00
[0,50]	[0,10]	832.50	911.68	894.80	890.90	889.42	8.64
[0,100]	[0,10]	667.90	753.07	734.14	729.61	726.84	11.02
[0,200]	[0,10]	572.10	662.88	646.34	634.68	630.89	20.81
[0,1000]	[0,10]	-683.70	-562.71	-583.11	-605.88	-611.49	28.21

Table 3: Continuous Relaxation Values

were demonstrated for the TVP. Future work consists of implementing our inequalities on the LOP, TSP, and CVRP. Relative to the TSP and based on the experience of [18], we feel that our inequalities (16) will be most useful when the auxiliary variables y_{ij} are naturally present within the formulation. As we noted earlier in this paper, [18] reported that the bounding restrictions (11) and 3-dicycle inequalities (12), together with (13), add considerable relaxation strength beyond that of the standard MTZ inequalities (6), but that the additional variables and constraints make the problem more computationally expensive to solve. However, when the variables y_{ij} are already present in the formulation, as with the enforcement of precedence restrictions, then these inequalities are useful. In the same way that inequalities (16) of TVP3 strengthen the inequalities (6) and (9) to the point where they are not explained in terms of the 3-dicycle inequalities (14), we expect increased strength by including (16) within the TSP. Computational experience, however, requires a specialized study that examines the numbers and structures of the precedence restrictions.

We believe that our conditional-logic for eliminating subtours will prove fruitful in more general scenarios than those described in Section 3.3, both when using the 3-dicycle inequalities (12) and the strengthened MTZ inequalities (9) and (10). An example of each setting is below.

As our first example, the paper of [21] uses linearized products of the form $x_{ik}y_{kj}$ to tighten the continuous relaxation of the TSP. If we choose to allow such products, then we can form more complex conditional statements as follows for each (i, j, k) distinct; $i, j, k \geq 2$.

$$\begin{array}{ll} \text{If } x_{ji} = 1 & \text{then } y_{ij} + y_{jk} + y_{ki} \leq 1. \\ \text{If } x_{ik}y_{kj} = 1 & \text{then } y_{ij} + y_{jk} + y_{ki} = 1. \\ \text{If } 1 - x_{ji} - x_{ik}y_{kj} = 1 & \text{then } y_{ij} + y_{jk} + y_{ki} \leq 2. \end{array}$$

Note that, unlike every previous application of our conditional logic, the above statements do not involve the variables u_j. Regardless, we have that each of the expressions x_{ji}, $x_{ik}y_{kj}$, and $1 - x_{ji} - x_{ik}y_{kj}$ found in the antecedents of these statements is binary for all feasible solutions to the TSP. Then we can again multiply each expression by its consequence and sum to obtain the following inequalities that are valid for the TSP.

$$y_{ij} + y_{jk} + y_{ki} + x_{ji} + x_{ik}y_{kj} \leq 2 \ \forall \ (i, j, k) \text{ distinct, } \ i, j, k \geq 2$$

These inequalities are a tightened version of the expressions (14) of [21] since the products $x_{ik}y_{kj}$ are nonnegative. For the TVP, we chose to not create such products because the additional linearized variables would have significantly increased the problem size.

Our second example deals with the quadratic traveling salesman problem (QTSP) [4, 5], and again involves the variables u_j. The QTSP is a generalization of the TSP that incurs an additional cost, for each distinct (i, j, k), if the salesman travels immediately from city i to city k and then immediately from city k to city j. (As with the TSP, we can assume without loss of generality that city 1 occurs first in the permutation.) Such costs necessitate quadratic expressions in the TSP decision variables of the form $x_{ik}x_{kj}$. With such expressions, we have for $n \geq 5$ that

$$x_{ij} + x_{ji} + \sum_{k \neq i,j} (x_{ik}x_{kj} + x_{jk}x_{ki}) \leq 1 \ \ \forall \ (i, j), \ i \neq j, \tag{30}$$

since $x_{ij} + x_{ji} = 1$ and $\sum_{k \neq i,j} (x_{ik}x_{kj} + x_{jk}x_{ki}) = 1$ represent that cities i and j are exactly one step apart, and exactly two steps apart, respectively, in the salesman's tour. Now, for $n \geq 5$, we can make the following conditional statements for each $(i, j), \ i, j \geq 2, \ i \neq j$.

$$\begin{array}{ll} \text{If } \sum_{k \neq 1,i,j} x_{ik}x_{kj} = 1 & \text{then } u_j - u_i = 2. \\ \text{If } x_{i1}x_{1j} = 1 & \text{then } u_j - u_i = 2 - n. \\ \text{If } \sum_{k \neq 1,i,j} x_{jk}x_{ki} = 1 & \text{then } u_j - u_i = -2. \\ \text{If } x_{j1}x_{1i} = 1 & \text{then } u_j - u_i = n - 2. \\ \text{If } x_{ij} = 1 & \text{then } u_j - u_i = 1. \\ \text{If } x_{ji} = 1 & \text{then } u_j - u_i = -1. \\ \text{If } 1 - x_{ij} - x_{ji} - \sum_{k \neq i,j} x_{ik}x_{kj} - \sum_{k \neq i,j} x_{jk}x_{ki} = 1 & \text{then } u_j - u_i \geq 3 - n. \end{array}$$

From (30), the expressions found in each of the seven antecedents is binary for all solutions to the QTSP. Then we can multiply each such expression by its associated consequence and surrogate to obtain the following quadratic inequality.

$$\begin{aligned} u_j - u_i \geq & (3-n) + (n-2)x_{ij} + (n-4)x_{ji} - x_{i1}x_{1j} + (2n-5)x_{j1}x_{1i} \\ & + (n-1)\sum_{k\neq 1,i,j} x_{ik}x_{kj} + (n-5)\sum_{k\neq 1,i,j} x_{jk}x_{ki} \ \ \forall\, (i,j),\ i,j\geq 2,\ i\neq j. \end{aligned}$$

For each $(i,j),\ i,j \geq 2,\ i \neq j$, the right side of the above inequality is greater than that of (9) by the quantity

$$1 - x_{ij} - x_{ji} - x_{i1}x_{1j} + (2n-5)x_{j1}x_{1i} + (n-1)\sum_{k\neq 1,i,j} x_{ik}x_{kj} + (n-5)\sum_{k\neq 1,i,j} x_{jk}x_{ki}.$$

This quantity is nonnegative since only $-x_{ij}$, $-x_{ji}$, and $-x_{i1}x_{1j}$ can be negative for $n \geq 5$, and we have that $x_{ij} + x_{ji} + x_{i1}x_{1j} \leq 1$ by (30). (Note that while the inequality is valid for $n = 5$, the last antecedent always holds false in this case.)

Continuing with our analysis of the QTSP, inequalities (10) can also be tightened. For $n \geq 5$, we can make the following conditional statements for each $j \geq 2$.

If $x_{1j} = 1$	then $u_j = 1$.
If $\sum_{k\neq 1,j} x_{1k}x_{kj} = 1$	then $u_j = 2$.
If $x_{j1} = 1$	then $u_j = n-1$.
If $\sum_{k\neq 1,j} x_{jk}x_{k1} = 1$	then $u_j = n-2$.
If $1 - x_{1j} - x_{j1} - \sum_{k\neq 1,j} x_{1k}x_{kj} - \sum_{k\neq 1,j} x_{jk}x_{k1} = 1$	then $3 \leq u_j \leq n-3$.

We now use (30) with $i = 1$ to note that the antecedents are all binary, so that we can multiply and surrogate to obtain the following two quadratic inequalities for each $j \geq 2$, upon separately considering the two possibilities in the last statement.

$$3 - 2x_{1j} + (n-4)x_{j1} - \sum_{k\neq 1,j} x_{1k}x_{kj} + (n-5)\sum_{k\neq 1,j} x_{jk}x_{k1} \leq u_j$$

$$u_j \leq (n-3) + (4-n)x_{1j} + 2x_{j1} + \sum_{k\neq 1,j} x_{jk}x_{k1} + (5-n)\sum_{k\neq 1,j} x_{1k}x_{kj}$$

Here, $3 \leq u_j$ and $u_j \leq n-3$ of the last statement give the first and second inequalities, respectively. (While these inequalities are valid for $n = 5$, the last antecedent always holds false in this case.) These inequalities tighten (10) by (30). To explain, for each $j \geq 2$, the left side of the first inequality is greater than the left expression of (10) by the quantity

$$1 - x_{1j} - x_{j1} - \sum_{k\neq 1,j} x_{1k}x_{kj} + (n-5)\sum_{k\neq 1,j} x_{jk}x_{k1},$$

and the right side of the second inequality is less than the right expression of (10) by this same amount. This quantity is nonnegative by (30) with $i = 1$, the nonnegativity of $\sum_{k\neq 1,j} x_{jk}x_{k1}$, and the assumption that $n \geq 5$.

Other tightenings can be obtained for the QTSP when the variables y_{ij} are also included within the formulation. Details are found in [3]. Future research is to further examine these tightenings and assess their computational merits.

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