ASYMPTOTICALLY OPTIMAL ALLOCATION POLICIES FOR TRANSPLANT QUEUEING SYSTEMS

AMIN KHADEMI* AND XIN LIU†

Abstract. We consider the problem of dynamic allocation of organs to patients in a transplant system. The system is modeled as a multi-class bipartite matching system, in which patients may die/delist or move between classes due to changes of their health status. We study a stochastic queueing control problem (QCP) with the control process governing the allocation of each arriving organ, and the objective of maximizing the expected total life years, which consists of both pre- and post-transplant years, of all population in the system during a finite time horizon. We first construct a deterministic control problem, referred to as the fluid control problem (FCP), and show that it serves as a performance upper bound for the QCP. We next develop an asymptotic framework, in which large scaled overloaded transplant systems are considered, and show that the fluid scaled QCP attains the FCP upper bound asymptotically. We then propose a simple priority type policy for the QCP based on the optimal solution of the FCP, and establish its asymptotic optimality through a scaling limit theorem. At last we conduct sensitivity analysis of the FCP with respect to the input parameters and functions to demonstrate the robustness of the proposed policy.

Key word. Organ transplant systems; Stochastic bipartite matching systems; Fluid approximations; Asymptotic optimality; Priority type policy; Sensitivity analysis.

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1. Introduction.

As of May 2018, 114,783 patients are waiting for transplants in the United States (cf. [35]). These patients are in the last stage of a disease and organ transplantation is a life-saving treatment. For example, heart transplantation is the only viable treatment for late-stage heart failure patients (cf. [30]). A deriving force for such a large waiting list is that the number of organs donated each year is significantly smaller than that of patients joining the waiting list. For example, at the end of year 2015, while 119,362 patients were on the waiting list, only 15,947 donors were recovered. This shortage of organ donation, which is predicted to continue, results in a substantial mortality for the patients in the waiting list, e.g., 20 patients die each day while waiting on the waiting list as studied in [37]. This imbalance of organ supply and demand raises an allocation question: which (eligible) patients should receive priority when an organ becomes available?

In order to address such a critical problem for the nation, in 1984, the U.S. Congress passed the National Organ Transplant Act (NOTA; P.L. 98-507), by which the Organ Procurement and Transplantation Network (OPTN) was established. The OPTN which is administrated by the United Network for Organ Sharing (UNOS) is responsible for maintaining a national registry for organs and managing organ allocation (cf. [26]). The NOTA enacted new rules to guarantee an efficient and fair allocation of donated organs via a priority rule that best matches the available organs to patients (cf. [12, 26]). Designing efficient and fair allocation rules considering heterogeneity of patient population and donated organs, as well as stochastic nature of the problem, among others, is extremely challenging and OPTN/UNOS has revised the organ allocation policies over time to overcome such challenges upon availability of more data and technology (see [36]). Despite all the efforts made, the waiting list has grown and as a result, in September 2016, the White House Office of Science and Technology issued a call to action to help reduce the waiting list for organ transplan-

*Department of Industrial Engineering, Clemson University (khademi@clemson.edu).
†School of Mathematical and Statistical Sciences, Clemson University (xliu9@clemson.edu).

tation, emphasizing that “ending the wait for organ transplants ... [is] some of what America can do” (see [34]).

The operations research community has studied the transplant system from a variety of standpoints, e.g., patient decision on accepting or declining an offered organ (cf. [11, 3]), addressing geographical disparity (cf. [6]), estimating the patient’s price of privacy (cf. [29]), designing point-based efficient and fair policies (cf. [8, 39, 1]), and kidney exchange (cf. [4]). In particular, [39] created a fluid model of kidney transplant queueing system to dynamically allocate kidneys to patients considering efficiency and fairness. In a series of papers, [31, 32, 33] studied the impacts of patient choice on waiting times, efficiency, and post-transplant information asymmetry using fluid models. Recently, [1] used an overloaded fluid queueing model to study the dynamic allocation of livers to patients. They considered a weighted objective of minimizing the pre-transplant deaths and maximizing the quality-adjusted life-years of the population. However, these studies did not formulate or analyze the stochastic control problem of dynamic allocation and settled to optimize the deterministic counterpart. In [38], the author considered a stochastic transplant queueing model, and focused on the asymptotic performance analysis under a randomized allocation policy. In this work, we formulate and analyze a stochastic control problem of dynamic matching of organs to patients and construct a simple/implementable matching policy that is asymptotically optimal under fluid scaling.

In the current work, we consider the problem of dynamic matching of heterogeneous organs to a heterogeneous patient population. We model the transplant system as a stochastic bipartite matching system, in which multi-class patients arrival to one side of the system, while multi-class organs arrival to the other side (see Figure 1). The arrival processes for both patients and organs are modeled by time-inhomogeneous Poisson processes. Furthermore, we assume that a patient may die/delist after an exponential amount of time, and the patient can also move to other classes due to the change of his/her health status after another exponential amount of time. Upon the arrival of an organ, the decision maker needs to decide immediately which (nonempty) patient queue should be offered the organ. Here we are interested in a stochastic queueing control problem (QCP) that maximizes an objective function which measures the total life year (pre- and post-transplant life years) of the entire population in a given finite time horizon. Our main contributions are summarized as follows: (1) We show that the fluid control problem (FCP), which is the deterministic counterpart of the stochastic QCP, naturally provides an upper bound for the QCP under any admissible policy. (2) We analyze the QCP in a large system regime, which is natural to transplant systems, and show that the fluid scaled QCP attains the FCP upper bound asymptotically. (3) We construct a policy for the QCP using the optimal solution of the FCP, and show that under an overloaded assumption, which is still natural to transplant systems, it is asymptotically optimal. An appealing feature of the constructed policy is that it is an index policy, which is aligned with OPTN/UNOS practice in considering priority rule policies. (4) We establish the robustness of the constructed policy for the FCP with respect to small perturbations of the input parameters and functions.

Dynamic matching in a two-sided market is also relevant to our study. However, the dynamic matching in transplant queueing systems is different and the policies developed do not apply. This is because in a transplant queueing systems, in addition to patient abandonment (patients may die or delist while waiting), the patients may change class (e.g., due to health deterioration), which is a key feature in the transplant queueing systems. Next, we briefly discuss the main differences between our work and related studies in dynamic matching. The paper [14] studied a multi-class queueing system where customers from one queue may match to those from (multiple) other queues to minimize a finite-horizon holding cost. A key difference between our work and this study is that in [14] the customers do not abandon or change class. In [19], the author studied the problem of matching of randomly arriving items to randomly arriving agents to minimize the probability of large deviations in expected waiting times of agents in overloaded systems. However, the agents do not abandon or change class in the queue. A similar argument holds true for the differences between our study and the literature on matching markets such as [5, 2, 17, 27]. In addition, the market thickness algorithms which are considered in matching markets do not apply to transplant queueing systems because the cold ischemic time (the time between chilling of an organ after its blood is cut off and the transplant time) is limited, e.g., it is four hours for heart/lung and one expects that the matching must be immediate. It is worth mentioning that [27] formulated a ridesharing queueing system in which heterogeneous customers request drivers with the objective to maximize the number of matches almost surely. They made the large market assumption, under which a deterministic counterpart of the stochastic control problem was studied, and based on the optimal solution to the deterministic control problem, an instantaneous probabilistic matching policy was constructed and shown to be asymptotically optimal. In fact, for time-inhomogeneous queueing systems, the asymptotic analysis is usually considered under a large market assumption with the fluid scaling (also see [7, 10, 25, 18]), and the FCP was shown to be the best performance bound for the original stochastic control problem asymptotically.

The rest of the paper is organized as follows: Section 2 formulates the stochastic QCP. Section 3 constructs the FCP and characterizes its optimal solutions. We analyze the QCP under a large system regime for overloaded transplant systems, and construct an asymptotically optimal matching policy in Section 4. In Section 5, we conduct a sensitivity analysis of the FCP under the proposed policy. Finally, Section 6 provides some numerical results, and Section 7 collects all the proofs.

The following notation will be used in the rest of the paper. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{R}^k$ denote the k-dimensional Euclidean space. Let $\mathbb{R}^K_+ = \{x \in \mathbb{R}^k : x_i \geq 0, i = 1, \ldots, k\}$. For $u \in \mathbb{R}^k$, its L_1 norm is denoted by $|u| = \sum_{i=1}^k |u_i|$, and its transpose is denoted by u', and we write $u > 0$ if $u_i > 0$ for all $i = 1, \ldots, k$. For a function f from $[0,\infty)$ to $\mathbb{R}^k$, define

$$|f|_t = \sup_{0\leq s\leq t} |f(s)|, \quad t \geq 0,$$

and

$$\int_s^t f(u)du = \left(\int_s^t f_1(u)du, \ldots, \int_s^t f_k(u)du\right)', \ [s,t] \subset [0,\infty).$$

For a differentiable function $f : [0,\infty) \to \mathbb{R}$, denote by $\dot{f}$ its derivative. We also use the following notation for the asymptotic behavior of nonnegative sequences $\{a_n\}$ and $\{b_n\}$:

$$a_n = o(b_n) \quad \text{if } a_n/b_n \to 0 \text{ as } n \to \infty,$$
$$a_n = O(b_n) \quad \text{if } a_n/b_n \leq C \text{ for all } n,$$

where C is a positive constant independent of n.

2. Problem Formulation. We consider an organ transplant system in which patients join a waiting list to receive organs. Upon arrival of an organ, a decision maker assigns it to a patient on the waiting list and a transplant is carried out. We categorize patients on the waiting list based on their different characteristics, such as age groups, blood types, and disease types, indexed by $i \in \mathbb{I} \equiv \{1, 2, \ldots, I\}$. So there are totally I numbers of patient queues. We assume that patients join queue i according to a Poisson process with time-varying rate $\{\lambda_i(t); t \geq 0\}$. Similarly, we categorize organs (e.g., hearts) based on different characteristics to reflect their quality, indexed by $h \in \mathbb{H} = \{1, 2, \ldots, H\}$, i.e., there are totally H numbers of organ groups (queues). The organs are assumed to arrive to queue h according to a Poisson process with time-varying rate $\{\mu_h(t); t \geq 0\}$. Poisson arrivals are standard assumptions in organ transplantation, e.g., see [6] for liver, and [33] for kidney. Also, we assume that if an organ is offered to a patient, it will be accepted by the patient, and if an organ becomes available and there is no patient in any queue, the organ becomes wasted. We further assume that a patient in queue i may die or delist (e.g., because of unsuitability for transplantation) after an exponential amount of time with rate d_i. The constant d_i can be interpreted as the patient health deterioration rate. Although pre-transplant mortality may depend on patients' waiting time, empirical studies show that isolating the effect of waiting time has a marginal impact on pre-transplant mortality. For example, using real data for heart transplantation, [16] showed that the pre-transplant mortality for patients who wait nine months (the expected waiting time to transplantation is seven months for heart) is increased only by 2%, on average, compared to no waiting. At last, patients can also move from one queue to another as their characteristics change (e.g., due to health deterioration). For simplicity, we assume that a patient moves from queue i to queue j after an exponential time with rate ρ_{ij}.

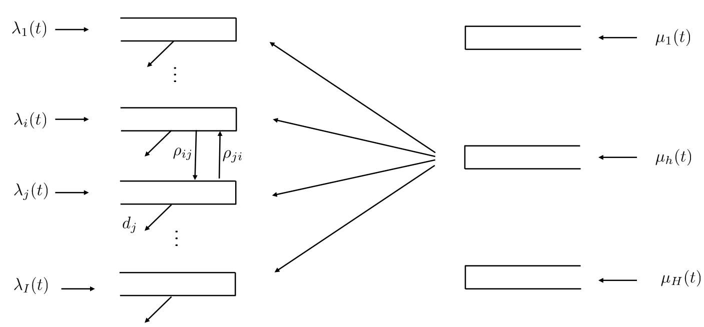

FIG. 1. *A schematic view of transplant queueing systems*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. All the random variables and stochastic processes in this section are assumed to be defined on this space. The expectation under $\mathbb{P}$ will be denoted by $\mathbb{E}$. Let $N_i, N_i^a, N_{ik}^b, i, k \in \mathbb{I}$, be independent unit rate Poisson processes that will be used to model the arrivals, abandonment, and class changes for patients. Independent of these Poisson processes, let A_h be a Poisson process with time-varying intensity $\{\mu_h(t); t \geq 0\}$, which represents the arrival process of organs to queue h. Denote by $X_i(t)$ the number of patients in queue i at time t, and $U_{hi}(t)$ the number of organs of type h assigned to patients in queue i up to time t. Then for $t \geq 0$, we have

(2.1)
$$X_i(t) = X_i(0) + N_i\left(\int_0^t \lambda_i(\tau)d\tau\right) - N_i^a\left(\int_0^t d_i X_i(\tau)d\tau\right) - \sum_{k\neq i} N_{ik}^b\left(\int_0^t \rho_{ik}X_i(\tau)d\tau\right) + \sum_{l\neq i} N_{li}^b\left(\int_0^t \rho_{li}X_l(\tau)d\tau\right) - \sum_h U_{hi}(t) \geq 0,$$

where $\{U_{hi}(t); t \geq 0\}$ is a counting process with nondecreasing sample paths satisfying

$$U_{hi}(0) = 0, \text{ and } \sum_{i\in\mathbb{I}} U_{hi}(t) \leq A_h(t), \text{ for all } t \geq 0, \tag{2.2}$$

and the initial value $X_i(0), i \in \mathbb{I}$ is independent of the Poisson processes N_i, N_i^a, N_{ik}^b, $i, k \in \mathbb{I}$, and $A_h, h \in \mathbb{H}$.

An allocation policy $\pi(t) = (\pi_{hi}(t))_{H\times I}, t \geq 0$, determines which patient queue is offered the arriving organ at time t. Each component $\pi_{hi}(t)$ equals to 0 or 1. When $\pi_{hi}(t) = 1$, then an organ of type h is arriving at time t and it is assigned to the first patient in queue i instantaneously. When $\pi_{hi}(t) = 0$, then either no organ of type h arrives at time t, or the arriving organ of type h at time t is assigned to a patient queue $j \neq i$, or the patient queues are all empty and the arriving organ of type h at time t is wasted. Thus for $h \in \mathbb{H}$ and $i \in \mathbb{I}$,

$$U_{hi}(t) = \sum_{k=1}^{A_h(t)} \mathbf{1}\{\pi_{hi}(\nu_{h,k}) = 1\}, \tag{2.3}$$

where $\{\nu_{h,k}\}_{k=1}^{\infty}$ are the arrival times of the organ arrival process A_h. An allocation policy $\pi \equiv \{\pi(t); t \geq 0\}$ is required not to anticipate the future. More precisely, define the filtration

$$\mathcal{F}_t = \sigma\left\{\left(X_i(s-), U_{hi}(s-), N_i\left(\int_{[0,s]} \lambda_i(\tau)d\tau\right), N_i^a\left(\int_{[0,s)} d_i X_i(\tau)d\tau\right),\right.\right.$$
$$\left.\left. N_{ik}^b\left(\int_{[0,s)} \rho_{ik}X_i(\tau)d\tau\right)\right), h \in \mathbb{H}, i, k \in \mathbb{I}, 0 \leq s \leq t\right\},$$

which represents all the information available to the decision maker at time t to allocate the arriving organ. A policy π is said to be *admissible* if it is non-anticipative, i.e., $\pi(t) \in \mathcal{F}_t$ for $t \geq 0$. The allocation process $U_{hi}(t), t \geq 0, h \in \mathbb{H}, i \in \mathbb{I}$, defined under an admissible policy π is called an admissible allocation process. Throughout the paper, we use π and U equivalently and intermittently when referring an admissible policy.

The control problem we consider is over a finite time horizon T. To ease notation, we introduce the following vectors and matrices to provide a matrix format formulation for the queueing control problem (QCP). Let $X = \{(X_1(t), \ldots, X_I(t)); t \in [0, T]\}$, $U_h = \{(U_{h1}(t), \ldots, U_{hI}(t))'; t \in [0, T]\}$, $\lambda = \{(\lambda_1(t), \ldots, \lambda_I(t))'; t \in [0, T]\}$, $\mu = \{(\mu_1(t), \ldots, \mu_H(t))'; t \in [0, T]\}$. Next let d be the $I \times I$ diagonal matrix with d_i being the i-th diagonal, $\rho = (\rho_{ij})_{I\times I}$ (note that $\rho_{ii} = 0$), and $\varrho = (\varrho_{ij})_{I\times I}$ be the $I\times I$ diagonal matrix with $\varrho_{ii} = \sum_{j\neq i} \rho_{ij}$ being the i-th diagonal. Denote by $\mathbf{1}$ the I-dimensional column vector of ones, and $\mathbf{e}_i$ the unit vector with one in the i-th coordinate. Finally, define the martrix-formed allocation process $U = \{(U_{hi}(t))_{H\times I}; t \in [0, T]\}$.

We consider a decision maker who seeks to maximize the total life years (pre- and post-transplant life years) of the entire population in a given horizon T. This objective function is a common measure of efficiency in organ transplantation literature, e.g., see [8] and references therein. Let α_{hi} be the expected post-transplant life years for patients in queue i transplanted by organs of type h; and β_i be the expected future life years of patients in queue i at the end of the planning horizon T. Define $\alpha_h = (\alpha_{h1}, \ldots, \alpha_{hI})'$, and $\beta = (\beta_1, \ldots, \beta_I)'$. Given the initial value $X(0) \in \mathbb{R}_+^I$, the objective is then to select an admissible allocation policy to maximize

$$\mathcal{J}(U; X(0), \mathcal{M}) \equiv \mathbb{E}\left\{\int_0^T \mathbf{1}'X(\tau)d\tau + \sum_{h\in\mathbb{H}} \alpha_h' U_h(T) + \beta' X(T)\right\}, \tag{2.4}$$

where $\mathcal{M} \equiv (\lambda, \mu, d, \rho, \alpha, \beta)$ that represents the information known to the decision maker, and in practice the decision maker can estimate $\mathcal{M}$ using historical data.

The QCP formulated above is intractable for direct analysis. In our asymptotic analysis, a deterministic fluid control problem (FCP) is developed, which can be solved explicitly. We then construct a policy for (2.4) by using the optimal solution of the FCP, and show that the proposed policy provides an upper bound, and is asymptotically optimal for (2.4).

3. Fluid Control Problem.

Fluid models describe the average behavior of stochastic systems, and provide good approximations for large scaled nonstationary stohastic systems (see e.g., [23, 22]). We introduce the following deterministic fluid control problem, which serves as an upper bound for the stochastic control problem developed in Section 2, and analyze its optimal solutions.

Let U be an admissible allocation process for (2.4). Taking expectations in (2.1) and (2.2) yields that for $t \in [0, T]$,

$$\mathbb{E}(X(t)) = \mathbb{E}(X(0)) + \int_0^t \lambda(\tau)d\tau - \int_0^t (d + \varrho - \rho')\mathbb{E}(X(\tau))d\tau - \sum_{h\in\mathbb{H}} \mathbb{E}(U_h(t)) \geq 0,$$

$$\mathbf{1}'\mathbb{E}(U_h(t)) \leq \int_0^t \mu_h(\tau)d\tau, \ \forall h \in \mathbb{H},$$

and

$$\mathbb{E}(U(0)) = 0, \text{ and } \mathbb{E}(U(t)) \text{ is nondecreasing in } t.$$

Now let $x_i(t) = \mathbb{E}(X_i(t))$ and $u_{hi}(t) = \mathbb{E}(U_{hi}(t))$ for $t \in [0, T]$. We define a deterministic control problem associated with the state process x_i and the control process u_{hi}. Let $x = \{(x_1(t), \ldots, x_I(t))';\ t \in [0, T]\}$, $u_h = \{(u_{h1}(t), \ldots, u_{hI}(t))'; t \in [0, T]\}$, and define the matrix function $u = \{(u_{hi}(t))_{H\times I};\ t \in [0, T]\}$.

Let $x_0 \in \mathbb{R}_+^I$. For the given data $\mathcal{M} \equiv (\lambda, \mu, d, \rho, \alpha, \beta)$, we consider the following deterministic control problem, which will be referred to as *the fluid control problem (FCP) associated with x_0 and $\mathcal{M}$.*

Definition 3.1. *Given $x_0 \in \mathbb{R}_+^I$ and $\mathcal{M}$, the FCP associated with x_0 and $\mathcal{M}$*

selects a $H \times I$ *matrix function* $u : [0, T] \to \mathbb{R}_+^{HI}$*, to maximize*

$$\bar{\mathcal{J}}(u; x_0, \mathcal{M}) \equiv \int_0^T \mathbf{1}' x(\tau) d\tau + \sum_{h \in \mathbb{H}} \alpha_h' u_h(T) + \beta' x(T) \tag{3.1a}$$

subject to the following conditions: For $t \in [0, T]$*,*

$$x(t) = x_0 + \int_0^t \lambda(\tau) d\tau - \int_0^t (d + \varrho - \rho') x(\tau) d\tau - \sum_{h \in \mathbb{H}} u_h(t) \geq 0, \tag{3.1b}$$

$$\mathbf{1}' u_h(t) \leq \int_0^t \mu_h(\tau) d\tau, \ \forall h \in \mathbb{H}, \tag{3.1c}$$

$$u(0) = 0, \ \text{and } u(t) \text{ is nondecreasing in } t. \tag{3.1d}$$

We assume that the arrival rate functions $\lambda : [0, T] \to \mathbb{R}_+^I$ and $\mu : [0, T] \to \mathbb{R}_+^H$ are continuous. Then there exists an optimal solution to the FCP associated with x_0 and $\mathcal{M}$ (see Theorem 3.1 in [15]). The optimal solutions to the FCP will be characterized in Proposition 3.3. Denote by $\bar{\mathcal{J}}^*(x_0, \mathcal{M})$ the optimal value of the FCP associated with x_0 and $\mathcal{M}$. One immediate result is that the FCP provides an upper bound for the original stochastic control problem (2.4), which is formalized in the following proposition.

Proposition 3.2. *Let* U *be an admissible allocation process for* (2.4)*. Then*

$$\mathcal{J}(U; X(0), \mathcal{M}) \leq \bar{\mathcal{J}}(\mathbb{E}(U); \mathbb{E}(X(0)), \mathcal{M}).$$

In particular,

$$\sup_U \mathcal{J}(U; X(0), \mathcal{M}) \leq \bar{\mathcal{J}}^*(\mathbb{E}(X(0)), \mathcal{M}),$$

where the supremum is taken over all the admissible allocation processes.

For the rest of the section, we study the optimal solutions to the FCP. We first provide an equivalent formulation for the FCP (3.1), which is in the standard form of optimal control theory. To that end, noting that the control process u is Lipstchitz continuous, for $i \in \mathbb{I}$, $h \in \mathbb{H}$, and $t \in [0, T]$, let $r_{hi}(t) = \dot{u}_{hi}(t)$, and it represents the rate at which organs of type h are assigned to patients of type i at time t. Let $r_h = \{(r_{h1}(t), \ldots, r_{hI}(t))'; t \in [0, T]\}$ and $r = \{(r_{hi}(t))_{H \times I}; t \in [0, T]\}$. For given $x_0 \in \mathbb{R}_+^I$ and $\mathcal{M} = (\lambda, \mu, d, \rho, \alpha, \beta)$, the formulation (3.1) is equivalent to the following linear optimal control with pure state constraints.

$$\max_r \bar{\mathcal{J}}(r; x_0, \mathcal{M}) \equiv \int_0^T \left(\mathbf{1}' x(t) + \sum_{h \in \mathbb{H}} \alpha_h' r_h(t) \right) dt + \beta' x(T) \tag{3.2a}$$

subject to the following conditions: For $t \in [0, T]$,

$$\dot{x}(t) = \lambda(t) - (d + \varrho - \rho') x(t) - \sum_{h \in \mathbb{H}} r_h(t), \tag{3.2b}$$

$$\mathbf{1}' r_h(t) \leq \mu_h(t), \ \forall h \in \mathbb{H}, \tag{3.2c}$$

$$r(t) \geq 0, \tag{3.2d}$$

$$x(t) \geq 0. \tag{3.2e}$$

The following proposition, adapted from [1], characterizes the structure of the optimal policy. The structure is further generalized to incorporate fairness constraints in

[16]. Let $k_i(t)$ be the shadow price (also known as the costate in optimal control) of the i-th constraint in (3.2b), which can be interpreted as a measure of the benefit the patients in queue i gain if not transplanted (see [1]). Let $k(t) = (k_1(t), k_2(t), \dots, k_I(t))'$ for $t \in [0, T]$.

PROPOSITION 3.3 ([1]). *A feasible allocation control $r \equiv \{r(t); t \in [0, T]\}$ is an optimal solution to the optimal control problem (3.2) with the corresponding state process $x \equiv \{x(t); t \in [0, T]\}$ if and only if there exist an I-dimensional shadow price $\{k(t); t \in [0, T]\}$, and an I-dimensional nonnegative piecewise absolutely continuous function $\{w(t); t \in [0, T]\}$ satisfying $w(t)'x(t) = 0$ such that*

$$\dot{k}(t) = (d + \varrho - \rho')k(t) - \mathbf{1} - w(t), \; t \in [0, T], \; k(T) = \beta, \tag{3.3}$$

and

$$r_h(t) \in \underset{z \in \mathbb{R}^I_+}{\arg\max} \{(\alpha_h - k(t))'z : \mathbf{1}'z \leq \mu_h(t)\}, \; h \in \mathbb{H}. \tag{3.4}$$

Remark 3.4. Given the shadow price $k(t)$, the optimization problem (3.4) is a linear knapsack problem with unit weights in the constraint for which the optimal solution is found by sorting the coefficients in the objective function. Therefore, the optimal policy for the the optimal control (3.2) is of priority rule type, i.e., if an organ of type h arrives, the decision maker sorts the queues in a descending order based on coefficients $\alpha_h - k(t)$, and assigns it to the nonempty queue with highest priority.

As interpreted in [1], the shadow price $k_i(t), i \in \mathbb{I}$, measures the future benefit of patients of class i without a transplant, and $\alpha_h - k(t)$ captures the difference in benefit with versus without transplantation with organs of type h. Thus the optimal policy is to assign the arriving organ to the patient queue with the highest transplantation benefit.

Calculating the priority rule depends on the value of shadows prices at each time. One way to calculate $k(t)$ is to discretize the optimal control problem (3.2) and because the discretized version is a linear program (LP), one may find the dual values computationally for each discretized time points. Another approach uses the fact that in an overloaded system where $x(t) > 0$, $k(t)$ is a solution to the following differential equation

$$\dot{k}(t) = (d + \varrho - \rho')k(t) - \mathbf{1},$$

with $k(T) = \beta$, which can be solved by using the matrix methods for linear ordinary differential equations [24].

Remark 3.5. In formulating the stochastic control problem, we assumed that each organ can be assigned to any class of patients. However, this assumption in the deterministic control problem is not restrictive because one may define a set of infeasible allocation, denoted by INF, such that $u_{hi}(t) = 0$ for all $t \in [0, T]$ for all organ-patient pair $(h, i) \in \text{INF}$. Following a similar steps in proving Proposition 3.3 shows that in the presence of such constraints the optimal policy is still a priority rule and in particular one solves the following knapsack

$$r_h(t) \in \underset{z \in \mathbb{R}^I_+}{\arg\max} \{(\alpha_h - k(t))'z : e'z \leq \mu_h(t), z_{hi} = 0, (h, i) \in \text{INF}\}, \quad h \in \mathbb{H}. \tag{3.5}$$

However, the shadow price $k(t)$ may change.

Remark 3.6. Recall that, in our model, patients are assumed to accept the offered organs. In order to incorporate different patient choices – accepting or declining the offer, for $h \in \mathbb{H}$, let P_h denote the $I \times I$ diagonal matrix with P_{hi}'s on its diagonal, where P_{hi} is the probability that an organ of type h is transplanted in queue i upon offer. Note that the cold ischemic time for an organ is limited, and so the number of times that an organ can be offered upon patient rejection is limited. Therefore, if we denote by D the number of times an organ can be offered and p_i^h the probability that a patient of type i will accept an organ of type h, then $P_{hi} = 1 - (1 - p_i^h)^D$. Therefore, the objective function of (3.2) changes to

$$\int_0^T \left(\mathbf{1}'x(t) + \sum_{h\in\mathbb{H}} \alpha_h' P_h r_h(t) \right) dt + \beta' x(T),$$

and the optimal solution solves

$$r_h(t) \in \underset{z\in\mathbb{R}_+^I}{\arg\max} \{(\alpha_h' P_h - k'(t))z : e'z \le \mu_h(t), , z_{hi} = 0, (h,i) \in \text{INF}\}, \quad h \in \mathbb{H}.$$

4. Asymptotic Optimality. This section develops an asymptotic framework, in which the suitably scaled stochastic control problem approaches a FCP as the system scale grows. We introduce a parameter n, which represents the system scale, and we will assume that the arrival rates of patients and organs are $\mathcal{O}(n)$. To make it precise, we consider a sequence of transplant queueing systems as described in Section 2, indexed by $n \in \mathbb{N}$. For the n-th system, we append a superscript n to all system processes, random variables, and parameters. In particular, on the space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, the processes $X_i^n(t)$ and $U_{hi}^n(t)$ are described as follows:

$$\begin{aligned} X_i^n(t) = X_i^n(0) &+ N_i^n \left(\int_0^t \lambda_i^n(\tau) d\tau \right) - N_i^{a,n} \left(\int_0^t d_i^n X_i^n(\tau) d\tau \right) \\ &- \sum_{k\ne i} N_{ik}^{b,n} \left(\int_0^t \rho_{ik}^n X_i^n(\tau) d\tau \right) + \sum_{l\ne i} N_{li}^{b,n} \left(\int_0^t \rho_{li}^n X_l^n(\tau) d\tau \right) \\ &- \sum_{h\in\mathbb{H}} U_{hi}^n(t) \ge 0. \end{aligned} \tag{4.1}$$

An allocation policy π^n is admissible if $\pi^n \in \mathcal{F}_t^n$, where

$$\begin{aligned} \mathcal{F}_t^n = \sigma \Bigg\{ \Bigg(& X_i^n(s-), U_{hi}^n(s-), N_i^n \left(\int_{[0,s]} \lambda_i^n(\tau) d\tau \right), N_i^{n,a} \left(\int_{[0,s)} d_i^n X_i^n(\tau) d\tau \right), \\ & N_{ik}^{n,b} \left(\int_{[0,s)} \rho_{ik}^n X_i^n(\tau) d\tau \right) \Bigg), h \in \mathbb{H}, i,k \in \mathbb{I}, 0 \le s \le t \Bigg\}. \end{aligned}$$

Under an admissible allocation policy π^n, the admissible allocation process is given by

$$U_{hi}^n(t) = \sum_{k=1}^{A_h^n(t)} \mathbf{1}\{\pi_{hi}^n(\nu_{h,k}^n) = 1\}, \ h \in \mathbb{H}, i \in \mathbb{I}, t \in [0,T]. \tag{4.2}$$

The objective is to choose π^n to maximize

$$\mathcal{J}^n(U^n; X^n(0), \mathcal{M}^n) \equiv \mathbb{E} \left\{ \int_0^T \mathbf{1}' X^n(\tau) d\tau + \sum_{h\in\mathbb{H}} (\alpha_h^n)' U_h^n(T) + (\beta^n)' X^n(T) \right\},$$

where $\mathcal{M}^n = (\lambda^n, \mu^n, d^n, \rho^n, \alpha^n, \beta^n)$.

We make the following assumptions on the model parameters and arrival rate functions.

ASSUMPTION 4.1 (Large scaled system).

(i) *For each $n \in \mathbb{N}$, the arrival rate funtions $\lambda^n : [0,T] \to \mathbb{R}^I_{\pm}$ and $\mu^n : [0,T] \to \mathbb{R}^H_+$ are continuous, and there exist continuous functions $\bar{\lambda} : [0,T] \to \mathbb{R}^I_+$ and $\bar{\mu} : [0,T] \to \mathbb{R}^H_+$ such that as $n \to \infty$,*

$$\left|\frac{\lambda^n}{n} - \bar{\lambda}\right|_T \to 0, \ \left|\frac{\mu^n}{n} - \bar{\mu}\right|_T \to 0.$$

(ii) *For each $n \in \mathbb{N}$, the parameters d^n, ρ^n, α^n, and β^n are nonnegative, and for $i, j \in \mathbb{I}$ and $h \in \mathbb{H}$, there exist nonnegative constants $\bar{\alpha}_{hi}, \bar{\beta}_i, \bar{d}_i, \bar{\rho}_{ij}$ such that as $n \to \infty$,*

$$\alpha^n_{hi} \to \bar{\alpha}_{hi}, \ \beta^n_i \to \bar{\beta}_i, \ d^n_i \to \bar{d}_i, \ \rho^n_{ij} \to \bar{\rho}_{ij}.$$

Remark 4.2. From the above assumptions, the arrival rates of patients and organs in the n-th system grow in the order of $\mathcal{O}(n)$. However, the abandonment and class change rates are of order $\mathcal{O}(1)$, which make them much smaller than the arrival rates especially when n becomes large. Nevertheless, the abandonment and class change processes are non-negligible in fluid limits as it is shown in the following analysis. Moreover, if the abandonment rates grow too fast (resp. too slowly), one can show that the fluid limit of the state process X^n becomes zero (resp. infinity); see [14, 20, 27, 23, 21] for similar types of scaling in queueing systems.

Under Assumption 4.1, the system grows to infinity as $n \to \infty$. To establish nontrivial limits, we consider the following fluid scaled processes and fluid scaled arrival rates: For $t \geq 0$, define

$$\bar{X}^n(t) = \frac{X^n(t)}{n}, \ \bar{U}^n(t) = \frac{U^n(t)}{n}, \ \bar{\lambda}^n(t) = \frac{1}{n}\lambda^n(t), \ \bar{\mu}^n(t) = \frac{1}{n}\mu^n(t).$$

Define the *fluid scaled* objective function by

$$\begin{aligned} \bar{\mathcal{J}}^n(U^n; X^n(0), \mathcal{M}^n) &= \frac{\mathcal{J}^n(U^n; X^n(0), \mathcal{M}^n)}{n} \\ &= \mathbb{E}\left\{\int_0^T \mathbf{1}'\bar{X}^n(\tau)d\tau + \sum_{h\in\mathbb{H}} (\alpha^n_h)'\bar{U}^n_h(T) + (\beta^n)'\bar{X}^n(T)\right\}. \end{aligned} \tag{4.3}$$

ASSUMPTION 4.3 (Initial condition). *For some deterministic $\bar{x}(0) \in \mathbb{R}^I_+ \setminus \{0\}$,*

$$\lim_{n\to\infty} \mathbb{E}[|\bar{X}^n(0) - \bar{x}(0)|] = 0.$$

Let $\bar{\mathcal{M}} \equiv (\bar{\lambda}, \bar{\mu}, \bar{d}, \bar{\rho}, \bar{\alpha}, \bar{\beta})$. Consider the FCP associated with $\bar{x}(0)$ and $\bar{\mathcal{M}}$, and let $\bar{\mathcal{J}}^*(\bar{x}(0), \bar{\mathcal{M}})$ denote its optimal value.

PROPOSITION 4.4. *Under Assumptions 4.1 and 4.3, for any admissible allocation policy π^n,*

$$\begin{aligned} \limsup_{n\to\infty} \bar{\mathcal{J}}^n(U^n; X^n(0), \mathcal{M}^n) &\leq \limsup_{n\to\infty} \frac{1}{n}\bar{\mathcal{J}}(\mathbb{E}(U^n); \mathbb{E}(X^n(0)), \mathcal{M}^n) \\ &\leq \bar{\mathcal{J}}^*(\bar{x}(0), \bar{\mathcal{M}}). \end{aligned}$$

Remark 4.5. From Proposition 3.2, for each $n \in \mathbb{N}$,

$$\bar{\mathcal{J}}^n(U^n; X^n(0), \mathcal{M}^n) = \frac{1}{n}\mathcal{J}^n(U^n; X^n(0), \mathcal{M}^n) \le \frac{1}{n}\bar{\mathcal{J}}(\mathbb{E}(U^n); \mathbb{E}(X^n(0)), \mathcal{M}^n),$$

which yields the first inequality. A key ingredient to prove the second inequality is to show that the expected fluid scaled allocation process $\mathbb{E}(\bar{U}^n)$ and its associated state process in the FCP associated with $\mathbb{E}(\bar{X}(0))$ and $\bar{\mathcal{M}}^n$ are uniformly bounded and Lipschitz continuous, and then to apply Arzela-Ascoli Theorem to obtain a convergent subsequence.

We next make the following overloaded assumption.

ASSUMPTION 4.6 (Overloaded system). *Let $\{x^*(t); t \in [0,T]\}$ be an optimal solution to the control problem (3.2) associated with $\bar{x}(0)$ and $\bar{\mathcal{M}}$. Then $x^*(t) > 0$ for $t \in [0,T]$.*

Remark 4.7. The transplant systems are usually heavily overloaded with a large number of patients waiting for organs. Similar assumptions are imposed in analyzing behaviors of such systems in, for example, [38, 1, 27]. In particular, [38] studied a transplant queueing system with constant arrival rates for both patients and organs, and considered a randomized allocation policy, under which the intensity for each patient queue can be defined using the mean allocation rates. And the overloaded condition, i.e., the intensity greater than one, was assumed for each patient queue. In [27], the authors assume that the fluid limit of the idle time process is zero, which implies that the system is overloaded.

Under Assumption 4.6, in the n-th system, it is possible that during a time period the allocation rate of organs to a patient queue is greater than its patient arrival rate, and is also possible that a patient queue becomes empty for a short time period.

An obvious sufficient condition for Assumption 4.6 is each patient arrival rate is strictly greater than the sum of organ arrival rates, i.e., $\int_0^t \lambda_i(s)ds > \int_0^t \sum_{h\in\mathbb{H}} \mu_h(s)ds$ for $t \in [0,T]$ and $i \in \mathbb{I}$. We also note that a necessary condition is $\int_0^t \sum_{i\in\mathbb{I}} \lambda_i(s)ds > \int_0^t \sum_{h\in\mathbb{H}} \mu_h(s)ds$ for $t \in [0,T]$.

Following Remark 3.4, we construct an optimal priority type allocation policy for the FCP associated with $\bar{x}_0$ and $\bar{\mathcal{M}}$. Under Assumption 4.6, the shadow price $\{k(t); t \in [0,T]\}$ is the unique solution to the linear system

$$\dot{k}(t) = (\bar{d} + \bar{\varrho} - \bar{\rho}')k(t) - \mathbf{1},\ t \in [0,T],\ k(T) = \bar{\beta}. \tag{4.4}$$

We rank the components of $\bar{\alpha}_h - k(t)$ in descending order, and use $Z^h(t) \equiv (Z^h_{\langle 1\rangle}(t), \ldots, Z^h_{\langle I\rangle}(t))$ to record the indices of the ranked components. The patient queue with index $Z^h_{\langle i\rangle}(t)$ has the i-th highest priority to receive the organ of type h that is arriving at time t. If $\bar{\alpha}_h - k(t)$ has tied components, we group the tied components, rank the groups in descending order, and within each group, we rank the components according to their indices in ascending order. For example, $Z^h_{\langle 1\rangle}(t)$ denotes the smallest index of the (tied) largest components. Noting that $k(\cdot)$ is smooth, the index function $Z^h(\cdot)$ changes values only finite times, and we can choose its values at the jump times properly to make it left continuous with right limits. For $i \in \mathbb{I}$, denote by $r^*_{hi}(t)$ the optimal allocation rate for organs of type h to patients in queue i at time t. Then for $t \in [0,T]$,

$$r^*_{h,Z^h_{\langle 1\rangle}(t)}(t) = \mu_h(t), \quad r^*_{h,i}(t) = 0,\ i \neq Z^h_{\langle 1\rangle}(t). \tag{4.5}$$

In particular, the optimal solution $r^* = \{(r^*_{hi}(t))_{H\times I}; t \in [0,T]\}$ is unique as the linear system (4.4) has a unique solution $k(\cdot)$. For $t \in [0,T]$, define the cumulative allocation process u^* to be

$$u^*(t) = \int_0^t r^*(s)ds, \ t \in [0,T]. \tag{4.6}$$

Now consider the n-th transplant queueing system. We propose the following allocation policy $\pi^{n,*}$. Recall that the organs of type h arrive according to independent time-varying Poisson processes A^n_h. Let $\nu^n_{h,m}$ denote the arrival time of the m-th organ of type h. At the arrival time $\nu^n_{h,m}$, the decision maker will consider a priority list $Z^h(\nu^n_{h,m}) = (Z^h_{\langle 1\rangle}(\nu^n_{h,m}), \ldots, Z^h_{\langle I\rangle}(\nu^n_{h,m}))$ such that the patient queue with index $Z^h_{\langle i\rangle}(\nu^n_{h,m})$ has the i-th highest priority to receive this organ of type h. The organ will be assigned to the nonempty patient queue with the highest priority; if all queues were empty, the organ is wasted. Therefore, under $\pi^{n,*}$, the cumulative number of organs of type h assigned to queue i under such allocation is given by

$$\begin{aligned} U^{n,*}_{hi}(t) = \sum_{m=1}^{A^n_h(t)} \sum_{j=1}^{I} \mathbf{1} \Bigg(& Z^h_{\langle j\rangle}(\nu^n_{h,m}) = i, X^n_i(\nu^n_{h,m}-) > 0, \\ & X^n_{Z^h_{\langle k\rangle}(\nu^n_{h,m})}(\nu^n_{h,m}-) = 0, k = 1,2,\ldots,j-1 \Bigg). \end{aligned} \tag{4.7}$$

Let $X^{n,*}$ denote the state process under the control process $U^{n,*}$ in the n-th system, and let x^* denote the state process under the optimal control u^* (4.6) in the FCP associated with $\bar{x}_0$ and $\bar{\mathcal{M}}$.

THEOREM 4.8. *Under Assumptions 4.1, 4.3, and 4.6, as $n \to \infty$,*

$$\left|(\bar{X}^{n,*}, \bar{U}^{n,*}) - (x^*, u^*)\right|_T \to 0, \ \textit{in probability.} \tag{4.8}$$

The next theorem shows that the upper bounds in Proposition 4.4 can be achieved under the proposed policy as $n \to \infty$.

THEOREM 4.9. *Under Assumptions 4.1, 4.3, and 4.6,*

$$\mathcal{J}^n(U^{n,*}; X^n(0), \mathcal{M}^n) = \bar{\mathcal{J}}^*(\mathbb{E}(X^n(0)), \mathcal{M}^n) + o(n), \ \textit{as } n \to \infty.$$

As a consequence of Theorem 4.9, the proposed policy $U^{n,*} = \{(U^{n,*}_{hi}(t))_{H\times I}; t \in [0,T]\}$ is *asymptotically optimal*, i.e., for an arbitrary sequence of admissible control processes $\{U^n\}_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \bar{\mathcal{J}}^n(U^{n,*}; X^n(0), \mathcal{M}^n) \geq \limsup_{n\to\infty} \bar{\mathcal{J}}^n(U^n; X^n(0), \mathcal{M}^n).$$

Remark 4.10. Consider an overloaded transplant system with rate functions and parameters given by $\mathcal{M}^n \equiv (\lambda^n, \mu^n, d^n, \rho^n, \alpha^n, \beta^n)$ satisfying Assumption 4.1. In practice, it means the arrival rates of both patients and organs are large, and they are much larger than the abandonment and class change parameters. Theorem 4.9 indicates that for such a system, our proposed policy $U^{n,*}$ is near optimal with error $o(n)$.

Remark 4.11. As mentioned in the section of introduction, fairness is another important objective in designing transplant allocation rules. One measure of fairness predominately used in the literature and practice is to minimize the pre-transplant mortality. That is, $\min \int_0^T \sum_{i\in\mathbb{I}} d_i X_i(t)dt$, where d_i is the death rate for patients of Class i and $X_i(t)$ denotes the number of patients of Class i waiting in the system at time t as formulated in (2.1) and (2.2). The corresponding FCP can be constructed similarly and the objective function becomes $\min \int_0^T \sum_{i\in\mathbb{I}} d_i x_i(t)dt$ with the same constraints as in the formulation (3.2). Similar arguments to the proof of Proposition 3.3 shows that the optimal policy is also of priority type. One can also construct asymptotic results similar to Theorems 4.8 and 4.9 for fairness objective. However, we do not show the details due to similarity of the analysis.

5. Sensitivity Analysis. Consider a transplant queueing system as described in Section 2, with rate functions and parameters $\mathcal{M} \equiv (\lambda, \mu, d, \rho, \alpha, \beta)$. In practice, one needs to estimate these parameters using historical data. Denote by $\hat{\mathcal{M}}^N \equiv (\hat{\lambda}^N, \hat{\mu}^N, \hat{d}^N, \hat{\rho}^N, \hat{\alpha}^N, \hat{\beta}^N)$ a sample estimator computed from a sample with size N. Assume that the estimator is consistent, i.e., as $N \to \infty$,

$$|\hat{\lambda}^N - \lambda|_T + |\hat{\mu}^N - \mu|_T + |\hat{d}^N - d| + |\hat{\rho}^N - \rho| + |\hat{\alpha}^N - \alpha| + |\hat{\beta}^N - \beta| \to 0, \text{ in probability.}$$

The priority allocation rule is constructed by these estimators. Specifically, a priority list is created based on the vector $\hat{\alpha}_h^N - k^N(t)$, where $k^N(t)$ satisfies the following linear system:

$$\dot{k}^N(t) = (\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)')k^N(t) - \mathbf{1},\ t \in [0,T],\ k^N(T) = \hat{\beta}^N.$$

A priority list $Z^{h,N}(t) = (Z^{h,N}_{\langle 1\rangle}(t), \dots, Z^{h,N}_{\langle I\rangle}(t))'$ at time t for organs of type h can be constructed based on the components of $\hat{\alpha}_h^N - k^N(t)$ following the way below (4.4). Define

$$u^{N,*}_{hi}(t) = \int_0^t \hat{\mu}^N_h(s)\mathbf{1}(Z^{h,N}_{\langle 1\rangle}(s) = i)ds,\ h \in \mathbb{H}, i \in \mathbb{I}, t \in [0,T].$$

Then $u^{N,*}$ is an optimal solution to the FCP associated with $\mathbb{E}(X(0))$ and $\hat{\mathcal{M}}^N$ if Assumption 4.6 holds.

The following proposition shows the robustness of the FCP with respect to the rate functions and parameters.

PROPOSITION 5.1. *Assume that the FCP associated with $\mathbb{E}(X(0))$ and $\hat{\mathcal{M}}^N$ satisfies Assumption 4.6 for all large N. Then*

$$\lim_{N\to\infty} \bar{\mathcal{J}}^*(\mathbb{E}(X(0)), \hat{\mathcal{M}}^N) = \bar{\mathcal{J}}^*(\mathbb{E}(X(0)), \mathcal{M}).$$

If further $\alpha_h - k(t)$ has a unique largest component for almost everywhere $t \in [0,T]$, then

$$\lim_{N\to\infty} |u^{N,*} - u^*|_T \to 0.$$

Similar to (4.7), one can construct an allocation process $U^{N,*}$ for the stochastic QCP (2.4) associated with $\hat{\mathcal{M}}^N$, that is for $h \in \mathbb{H}$ and $i \in \mathbb{I}$,

$$U^{N,*}_{hi}(t) = \sum_{m=1}^{A_h(t)} \sum_{j=1}^{I} \mathbf{1}\left(Z^{h,N}_{\langle j\rangle}(\nu_{h,m}) = i, X^N_i(\nu_{h,m}-) > 0, \right.$$
$$\left. X^N_{Z^{h,N}_{\langle k\rangle}(\nu_{h,m})}(\nu_{h,m}-) = 0, k = 1, 2, \dots, j-1 \right),$$

where X^N is the corresponding state process with $X^N(0) = X(0)$ and $\mathcal{M}^N$. In view of Proposition 5.1 and Theorem 4.9, when

(i) the transplant system is overloaded,

(ii) both $\hat{\lambda}^N$ and $\hat{\mu}^N$ are much larger than $\hat{d}^N, \hat{\rho}^N, \hat{\alpha}^N$, and $\hat{\beta}^N$,

we have

$$\mathcal{J}(U^{N,*}; X(0), \hat{\mathcal{M}}^N) = \bar{\mathcal{J}}^*(\mathbb{E}(X(0)), \mathcal{M}) + o\left(|\hat{\lambda}^N|_T\right), \tag{5.1}$$

which says that the proposed allocation policy is near optimal with error $o(|\hat{\lambda}^N|_T)$.

6. Numerical Results. In this section, we provide a numerical example to illustrate the behavior of the FCP upper bound derived in Proposition 3.2, and the performance of the proposed policies via simulations. In particular, we design a transplant system, where there are four patient groups and two organ types. The horizon is set to $T = 365$ days. The patient arrival rates are $\lambda = (1, 3, 2, 1)'$ per day, and organ arrival rates are $\mu = (1, 2)'$ per day. The death rates (per day) for patient groups are $d = diag(0.2, 0.04, 0.02, 0.2)$ and final rewards for patients on the waiting list at the end of the horizon are $\beta = (20, 17, 16, 16)'$. We set rewards from transplanting with organ type 1 as $\alpha_1 = (10, 50, 100, 10)$ and those with organ type 2 as $\alpha_2 = (10, 50, 100, 10)'$. The class-change rate matrix ρ is presented in Table 1. We set the initial population to be $X(0) = (10, 15, 20, 25)'$.

We solve the FCP (3.2) by discretizing time. By doing so the FCP (3.2) becomes a linear program (LP), which is easy to solve. Moreover, since optimal control (3.2) satisfies conditions in a corollary for Theorem 3 in [9], as the number of discretization steps increases, the optimal value of the discretized version of the FCP converges to the optimal value of the FCP.

Table 1
Patient class change probabilities

Patient group	1	2	3	4
1	-	0.014	0.0006	0.002
2	0.0043	-	0.0008	0.0021
3	0.0001	0.0007	-	0.0014
4	0.00019	0.00011	0.00028	-

To construct the nth system in the asymptotic framework, we set $\lambda^n = n\lambda + \sqrt{n}$, $\mu^n = n\mu + \sqrt{n}$, $\rho^n = \rho$, $\alpha^n = \alpha$, $\beta^n = \beta$. Then, we plug these quantities into the FCP (3.2) and solve its discretized version, which is a linear program, and report its objective function divided by n. Furthermore, we simulate the stochastic system using our proposed policy to observe its performance. To that end, for each given n, we find the shadow price $k^n(t)$ at each time t by extracting the optimal dual value for the corresponding constraint of the LP. Note that if the state is positive at all time, one can solve the system of linear differential equations in (4.4) to calculate the shadow prices for all time. However, the approach we use in this section is more general and can be used for the cases that the state becomes zero over a time interval. Then, at each time t, we construct the priority list $Z^{h,n}$ for the organs of type h based on $\alpha_h^n - k^n(t)$ and allocate the arrived organ to the nonempty patient group with the highest priority. In each patient group, we follow a first-come first-serve discipline. If all the patient groups was empty, the organ becomes wasted. In the stochastic simulation, we follow the above policy and find the expected life years and divide the objective by n. The

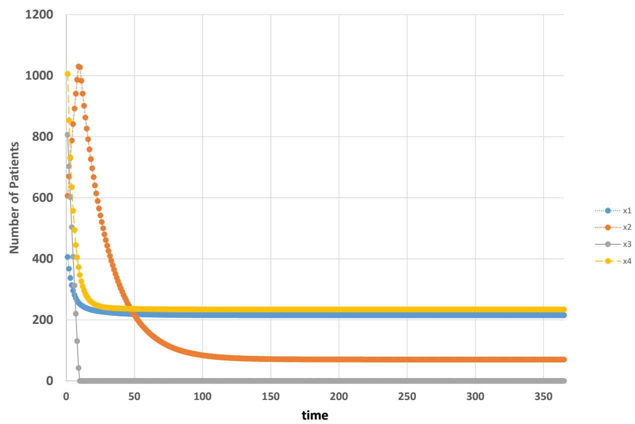

FIG. 2. *The four curves represent the queue sizes in the four classes of patients. In particular, $X_3(\cdot)$ stays at 0 after a short time period.*

state process $\{X(t) = (X_1(t), X_2(t), X_3(t), X_4(t)); t \geq 0\}$ is given in Figure 2. We note that $X_3(\cdot)$ stays at 0 after a short time period, which voids Assumption 4.6. However, using the above method, we can compute $k^n(t), t \in [0, T]$. In Figure 3, the upper dashed red curve FCP(n)/n shows the fluid scaled optimal value of the FCP associated with $nX(0) + \sqrt{n}$ and $\mathcal{M}^n = (\lambda^n, \mu^n, d^n, \rho^n, \alpha^n, \beta^n)$; and the lower solid black curve Simulation(n)/n shows the fluid scaled value of the QCP associated with $nX(0) + \sqrt{n}$ and $\mathcal{M}^n = (\lambda^n, \mu^n, d^n, \rho^n, \alpha^n, \beta^n)$ under the proposed policy from simulations. As can be seen, the result from simulations fluctuates especially for small n and disappears for large n, and the two curves are getting close as n increases.

7. Proofs.

This section presents the proofs of propositions and theorems. We first show the C-tightness and uniform integrability of $(\bar{X}^n, \bar{U}^n)$ under any admissible allocation policy π^n. Next the asymptotic results of $(\bar{X}^{n,*}, \bar{U}^{n,*})$ under the proposed admissible allocation policy $\pi^{n,*}$ are provided. At last, we complete all the proofs of propositions and theorems in Sections 4 and 5.

PROPOSITION 7.1. *Under any admissible allocation policy π^n, $\{(\bar{X}^n, \bar{U}^n)\}_{n\in\mathbb{N}}$ is C-tight and uniformly integrable.*

The proof of Proposition 7.1 is provided in the supplementary materials, and it is divided into Lemma SM0.1 (proving the stochastic boundedness), Lemmas SM0.2 and SM0.3 (proving the convergence of the continuity modulus), and Lemma SM0.4 (establishing the uniform integrability).

We next consider the proposed allocation policy $\pi^{n,*}$. We first consider a modification of $\pi^{n,*}$ and $\{U^{n,*}_{hi}(t); t \in [0, T]\}$ defined in (4.7). Let $\tilde{\pi}^n$ denote this modified allocation policy, under which upon arrival of an organ of type h at time t, it will be assigned to the patient queue with index $Z^h_{\langle 1\rangle}(t)$ if this patient queue is nonempty, and otherwise it will be wasted if it is empty. Denote by $\{\mathcal{U}^n_{hi}(t); t \in [0, T]\}$ and

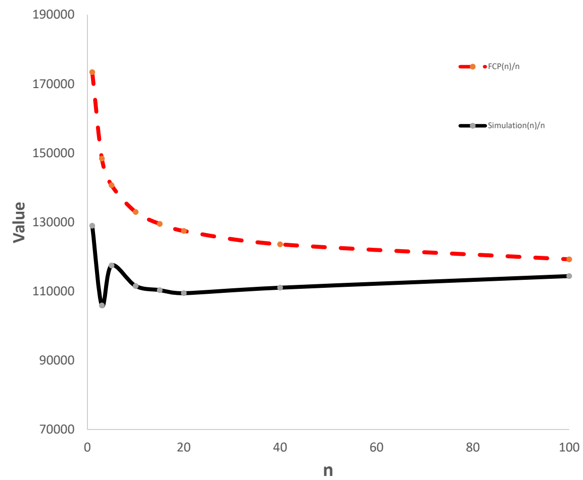

FIG. 3. *The upper dashed red curve gives the fluid scaled optimal value of the FCP associated with $nX(0)+\sqrt{n}$ and $\mathcal{M}^n = (\lambda^n, \mu^n, d^n, \rho^n, \alpha^n, \beta^n)$, and the lower solid black curve gives the fluid scaled value of the QCP with $nX(0)+\sqrt{n}$ and $\mathcal{M}^n$ under the proposed allocation policy.*

$\{\mathcal{X}^n(t) = (\mathcal{X}_1^n(t), \ldots, \mathcal{X}_I^n(t))'; t \in [0,T]\}$ the allocation and state processes under $\tilde{\pi}^n$. Here $\mathcal{X}^n(0) = X^n(0)$. We have for $h \in \mathbb{H}, i \in \mathbb{I}$, and $t \in [0,T]$,

$$\mathcal{U}_{hi}^n(t) = \sum_{m=1}^{A_h^n(t)} \mathbf{1}\left(Z_{\langle 1\rangle}^h(\nu_{h,m}^n) = i,\ \mathcal{X}_i^n(\nu_{h,m}^n-) > 0\right). \tag{7.1}$$

LEMMA 7.2. *For $t \in [0,T]$ and $i \in \mathbb{I}$, as $n \to \infty$, in probability,*

$$\left|\int_0^t \mathbf{1}(Z_{\langle 1\rangle}^h(\tau) = i, \bar{\mathcal{X}}_i^n(\tau-) > 0) d\bar{A}_h^n(\tau) - \int_0^t \mathbf{1}(Z_{\langle 1\rangle}^h(\tau) = i, \bar{\mathcal{X}}_i^n(\tau) > 0)\bar{\mu}_h(\tau) d\tau\right| \to 0.$$

The proof of the above lemma can be found in the supplementary materials.

PROPOSITION 7.3. *Under Assumptions 4.3, 4.1 and 4.6, as $n \to \infty$, in probability,*

$$\left|(\bar{\mathcal{X}}^n, \bar{\mathcal{U}}^n) - (x^*, u^*)\right|_T \to 0. \tag{7.2}$$

Proof. Under Assumption 4.6, the shadow price process $\{k(t); t \in [0,T]\}$ is differentiable, and consequently, for each $h \in \mathbb{H}$, the index process $\{Z^h(t); t \in [0,T]\}$ has a finite number of jumps for each of its components. We partition $[0,T]$ into M intervals $[0,t_1], (t_1,t_2], \ldots, (t_{M-1},T]$ such that during each subinterval, $Z^h(t)$ is constant. Let $t_0 = 0$. It suffices to focus on each subinterval $(t_i, t_{i+1}]$, and show that $\sup_{t\in[t_i,t_{i+1})} \left|(\bar{\mathcal{X}}^n(t), \bar{\mathcal{U}}^n(t)) - (x^*(t), u^*(t))\right| \to 0$, in probability. Let's focus on the subinterval $[0,t_1]$. Denote by $Z_{\langle 1\rangle}^h$ the index of the patient queue of highest priority for the organs of class h during the subinterval $[0,t_1)$. Using Lemma 7.2, for $t\in[0,t_1]$,

$$\left|\sum_{h\in\mathbb{H}}\bar{\mathcal{U}}^n_{hi}(t)-\sum_{h\in\mathbb{H}}\int_0^t\bar{\mu}_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\tau)>0)d\tau\right|$$
$$=\left|\frac{1}{n}\sum_{h\in\mathbb{H}}\sum_{m=1}^{A^n_h(t)}\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\nu^n_{h,m}-)>0)-\sum_{h\in\mathbb{H}}\int_0^t\bar{\mu}_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\tau)>0)d\tau\right|$$
$$\le\sum_{h\in\mathbb{H}}\left|\int_0^t\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\tau-)>0)d\bar{A}^n_h(\tau)-\int_0^t\bar{\mu}_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\tau)>0)d\tau\right|$$
$$\to 0.$$

Define the following centered process: For $t\in[0,t_1)$,

$$\begin{aligned}\bar{M}^n_i(t)=\bar{N}^n_i\left(\int_0^t\bar{\lambda}^n_i(\tau)d\tau\right)-\int_0^t\bar{\lambda}^n_i(\tau)d\tau-\bar{N}^{a,n}_i\left(\int_0^t d^n_i\bar{\mathcal{X}}^n_i(\tau)d\tau\right)+\int_0^t d^n_i\bar{\mathcal{X}}^n_i(\tau)d\tau\\ -\sum_{k\neq i}\left(\bar{N}^{b,n}_{ik}\left(\int_0^t\rho^n_{ik}\bar{\mathcal{X}}^n_i(\tau)d\tau\right)-\int_0^t\rho^n_{ik}\bar{\mathcal{X}}^n_i(\tau)d\tau\right)\\ +\sum_{l\neq i}\left(\bar{N}^{b,n}_{li}\left(\int_0^t\rho^n_{li}\bar{\mathcal{X}}^n_l(\tau)d\tau\right)-\int_0^t\rho^n_{li}\bar{\mathcal{X}}^n_l(\tau)d\tau\right)\\ +\sum_{h\in\mathbb{H}}\bar{\mathcal{U}}^n_{hi}(t)-\sum_{h\in\mathbb{H}}\int_0^t\bar{\mu}_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\tau)>0)d\tau.\end{aligned}$$

From Proposition 7.1, $\bar{\mathcal{X}}^n$ is C-tight. By the functional law of large numbers, $\bar{N}^n_i(\cdot)$, $\bar{N}^{b,n}_{ij}(\cdot), i,j\in\mathbb{I}$, all converge to the identity map $\iota:[0,\infty)\to[0,\infty)$. Finally, by continuous mapping theorem, $\bar{M}^n\Rightarrow 0$.

We next note that for $t\in[0,t_1]$,

$$\begin{aligned}\bar{\mathcal{X}}^n_i(t)&=\bar{X}^n_i(0)+\bar{M}^n_i(t)-\int_0^t d^n_i\bar{\mathcal{X}}^n_i(\tau)d\tau-\sum_{k\neq i}\int_0^t\rho^n_{ik}\bar{\mathcal{X}}^n_i(\tau)d\tau+\sum_{l\neq i}\int_0^t\rho^n_{li}\bar{\mathcal{X}}^n_l(\tau)d\tau\\ &\quad+\int_0^t\bar{\lambda}^n_i(\tau)d\tau-\sum_{h\in\mathbb{H}}\int_0^t\bar{\mu}^n_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i,\bar{\mathcal{X}}^n_i(\tau)>0)d\tau\\ &=\bar{X}^n_i(0)+\bar{M}^n_i(t)-\int_0^t d^n_i\bar{\mathcal{X}}^n_i(\tau)d\tau-\sum_{k\neq i}\int_0^t\rho^n_{ik}\bar{\mathcal{X}}^n_i(\tau)d\tau+\sum_{l\neq i}\int_0^t\rho^n_{li}\bar{\mathcal{X}}^n_l(\tau)d\tau\\ &\quad+\int_0^t\left(\bar{\lambda}^n_i(\tau)-\sum_{h\in\mathbb{H}}\bar{\mu}^n_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i)\right)d\tau\\ &\quad+\int_0^t\left(\sum_{h\in\mathbb{H}}\bar{\mu}^n_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i)\right)\mathbf{1}(\bar{\mathcal{X}}^n_i(\tau)=0)d\tau.\end{aligned}$$

Let

$$\bar{C}^n_i(t)=\int_0^t\left(\sum_{h\in\mathbb{H}}\bar{\mu}_h(\tau)\mathbf{1}(Z^h_{\langle 1\rangle}=i)\right)\mathbf{1}(\bar{\mathcal{X}}^n_i(\tau)=0)d\tau.$$

We note that $\bar{C}_i^n(0) = 0$, $\bar{C}_i^n(t)$ is nondecreasing in t, and $\int_0^T \bar{\mathcal{X}}_i^n(\tau) d\bar{C}_i^n(\tau) = 0$. Next let

$$\bar{Z}_i^n(t) = \bar{M}_i^n(t) - \int_0^t d_i^n \bar{\mathcal{X}}_i^n(\tau) d\tau - \sum_{k\neq i} \int_0^t \rho_{ik}^n \bar{\mathcal{X}}_i^n(\tau) d\tau + \sum_{l\neq i} \int_0^t \rho_{li}^n \bar{\mathcal{X}}_l^n(\tau) d\tau$$
$$+ \int_0^t \left(\bar{\lambda}_i^n(\tau) - \sum_{h\in\mathbb{H}} \bar{\mu}_h^n(\tau) \mathbf{1}(Z_{\langle 1\rangle}^h = i) \right) d\tau.$$

From [28], $(\bar{\mathcal{X}}^n, \bar{C}^n) = (\Phi, \Psi)(\bar{Z}^n)$, where Φ and Ψ are Lipschitz continuous regulator mappings. The C-tightness of $\bar{Z}^n$ yields that $(\bar{\mathcal{X}}^n, \bar{C}^n, \bar{Z}^n)$ is C-tight. Denote by $(\bar{x}, \bar{c}, \bar{z})$ its weak limit. Then

$$\bar{x}_i(t) = \bar{x}_i(0) + \bar{z}_i(t) + \bar{c}_i(t),$$

and

$$\bar{z}_i(t) = -\int_0^t [(\bar{d} + \bar{\varrho} - \bar{\rho}')\bar{x}(\tau)]_i d\tau + \int_0^t \left(\bar{\lambda}_i(\tau) - \sum_{h\in\mathbb{H}} \bar{\mu}_h(\tau) \mathbf{1}(Z_{\langle 1\rangle}^h = i) \right) d\tau.$$

Define $\xi \in [0, t_1]$ to be the first time $\bar{x}(\cdot)$ attains the boundary of $\mathbb{R}_+^I$. We will show that ξ doesn't exist, and consequently, $\bar{x}_i(t) > 0$ for all $t \in [0, t_1]$ and $i \in \mathbb{I}$. If there exists such a ξ, then for $t \in [0, \xi]$, $\bar{c}(t) = 0$, and for $i \in \mathbb{I}$,

$$\bar{x}_i(t) = \bar{x}_i(0) - \int_0^t [(\bar{d} + \bar{\varrho} - \bar{\rho}')\bar{x}(\tau)]_i d\tau + \int_0^t \left(\bar{\lambda}_i(\tau) - \sum_{h\in\mathbb{H}} \bar{\mu}_h(\tau) \mathbf{1}(Z_{\langle 1\rangle}^h = i) \right) d\tau.$$

We note that $\bar{x}(\cdot)$ is an optimal solution to the control problem (3.2) associated with the data $\bar{\mathcal{M}}$ over the time interval $[0, \xi]$. According to Assumption 4.6, $\bar{x}(t) > 0$ for $t \in [0, \xi]$, which contradicts that $\bar{x}(\xi) = 0$. Hence $\bar{x}_i(t) > 0$ and $\bar{c}_i(t) = 0$ for all $t \in [0, t_1]$ and $i \in \mathbb{I}$, and $\bar{x}(\cdot)$ is an optimal solution to the control problem (3.2) associated with the data $\bar{\mathcal{M}}$ over the time interval $[0, t_1]$. It follows now that $\sup_{0\le t\le t_1} \left|(\bar{\mathcal{X}}^n(t), \bar{\mathcal{U}}^n(t)) - (x^*(t), u^*(t))\right| \to 0$ in probability. The analysis of the convergence over other subintervals will be similar, and we omit the details. ∎

LEMMA 7.4. *Under Assumptions 4.1 and 4.6, for all $t \in [0, T]$, as $n \to \infty$, in probability,*

$$\left|(\bar{\mathcal{X}}^n, \bar{\mathcal{U}}^n) - (\bar{X}^{n,*}, \bar{U}^{n,*})\right|_T \to 0. \tag{7.3}$$

Proof. By the continuous mapping theorem and Proposition 7.3, we have

$$\inf_{0\le t\le T} \bar{\mathcal{X}}^n(t) \to \inf_{0\le t\le T} x^*(t), \text{ in probability.} \tag{7.4}$$

Noting that $x_i^*(t)$ is absolutely continuous for $t \in [0, T]$, which implies that

$$\inf_{0\le t\le T} x_i^*(t) > 0.$$

For $\epsilon \in (0, \min_{i\in\mathbb{I}} \inf_{0\le t\le T} x_i^*(t))$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\inf_{0\le t\le T} \bar{\mathcal{X}}_i^n(t) \ge \inf_{0\le t\le T} x_i^*(t) - \epsilon \text{ for all } i \in \mathbb{I} \right) = 1.$$

Fix an $\epsilon \in (0, \min_{i\in\mathbb{I}} \inf_{0\le t\le T} x_i^*(t))$, and let $A_n = \{\inf_{0\le t\le T} \mathcal{X}_i^n(t) \ge \inf_{0\le t\le T} x_i^*(t) - \epsilon$ for all $i \in \mathbb{I}\}$. On A_n, $\mathcal{X}^n(t) > 0$ for all $t \in [0,T]$, and the modified policy $\mathcal{U}_{hi}^n(t) = \sum_{m=1}^{A_h^n(t)} \mathbf{1}(Z_{\langle 1\rangle}^h(\nu_{h,m}^n) = i)$. Recall that, under the proposed policy $U^{n,*}$, for $t \in [0,T]$,

$$X_i^{n,*}(t) = X_i^n(0) + N_i^n\left(\int_0^t \lambda_i^n(\tau)d\tau\right) - N_i^{a,n}\left(\int_0^t d_i^n X_i^{n,*}(\tau)d\tau\right)$$
$$- \sum_{k\ne i} N_{ik}^{b,n}\left(\int_0^t \rho_{ik}^n X_i^{n,*}(\tau)d\tau\right) + \sum_{l\ne i} N_{li}^{b,n}\left(\int_0^t \rho_{li}^n X_l^{n,*}(\tau)d\tau\right) - \sum_{h\in\mathbb{H}} U_{hi}^{n,*}(t),$$

and

$$U_{hi}^{n,*}(t) = \sum_{m=1}^{A_h^n(t)} \sum_{j=1}^{I} \mathbf{1}\left(Z_{\langle j\rangle}^h(\nu_{h,m}^n) = i, X_i^{n,*}(\nu_{h,m}^n -) > 0, X_{Z_{\langle k\rangle}^h(\nu_{h,m}^n)}^{n,*}(\nu_{h,m}^n -) = 0,\right.$$
$$\left. k = 1, 2, \ldots, j-1 \right).$$

Let β denote the first time the process $X^{n,*}$ reaches the boundary of $\mathbb{R}_+^I$. For $t \in [0,\beta)$, we have $X^{n,*}(t) > 0$, and

$$U_{hi}^{n,*}(t) = \sum_{m=1}^{A_h^n(t)} \mathbf{1}(Z_{\langle 1\rangle}^h(\nu_{h,m}^n) = i).$$

It now follows that for all $t \in [0,\beta)$,

$$U^{n,*}(t) = \mathcal{U}^n(t), \quad X^{n,*}(t) = \mathcal{X}^n(t).$$

At the time β, if an organ of type h arrives, it will be assigned to the patient queue with index $Z_{\langle 1\rangle}^h(\beta)$ under both the modified allocation policy $\tilde{\pi}^n$ and the policy $\pi^{n,*}$. Thus we have

$$U^{n,*}(\beta) = \mathcal{U}^n(\beta), \quad X^{n,*}(\beta) = \mathcal{X}^n(\beta).$$

Now on A_n, $X^{n,*}(\beta) = \mathcal{X}^n(\beta) > 0$, which contradicts the definition of β. Hence on A_n, $X^{n,*}(t) > 0$ for all $t \in [0,T]$, which says that $U_{hi}^{n,*}(t) = \mathcal{U}_{hi}^n(t)$ and $X^{n,*}(t) = \mathcal{X}^n(t)$ for all $t \in [0,T]$. Noting that $\mathbb{P}(A_n) \to 1$, the lemma follows. ∎

We now provide the complete proofs of all theorems and propositions.

Proof of Proposition 3.2. For an admissible process $\{U(t); t \in [0,T]\}$ defined in (2.3), let $u(t) = \mathbb{E}(U(t))$ and $x(t) = \mathbb{E}(X(t))$. Then (x,u) satisfies the constraints of the FCP, thus a feasible solution. Also, the objective function of the stochastic QCP is linear in $X(t)$ and $U(t)$, and then the objective function of (2.4) and (3.1) become equivalent. Therefore,

$$\mathcal{J}(U; X(0), \mathcal{M}) \le \bar{\mathcal{J}}(\mathbb{E}(U); \mathbb{E}(X(0)), \mathcal{M}).$$

Taking supremum from both sides yields

$$\sup_U \mathcal{J}(U; X(0), \mathcal{M}) \le \bar{\mathcal{J}}^*(\mathbb{E}(X(0)), \mathcal{M}),$$

which completes the proof. ∎

Proof of Proposition 4.4. It suffices to show that

$$\limsup_{n\to\infty} \frac{1}{n}\bar{\mathcal{J}}(\mathbb{E}(U^n);\mathbb{E}(X^n(0)),\mathcal{M}^n) \le \bar{\mathcal{J}}^*(\bar{x}(0),\bar{\mathcal{M}}). \tag{7.5}$$

For $t\in[0,T]$, let $u^n(t)=\mathbb{E}(U^n(t))$ and $x^n(t)=\mathbb{E}(X^n(t))$. From the proof of Proposition 3.2, u^n is an admissible control for the FCP associated with $x^n(0)$ and $\mathcal{M}^n$. Now define the fluid scaled processes $\bar{u}^n(t)=u^n(t)/n$ and $\bar{x}^n(t)=x^n(t)/n$. We note that $\bar{u}^n$ is an admissible control for the FCP associated with $\bar{x}^n(0)$ and $\bar{\mathcal{M}}^n\equiv(\bar{\lambda}^n,\bar{\mu}^n,d^n,\rho^n,\alpha^n,\beta^n)$. Assumption 4.1 (i) yields that

$$\sup_n|\bar{\lambda}^n|_T<\infty,\ \sup_n|\bar{\mu}^n|_T<\infty,$$

which, combining with Assumption 4.1(ii), implies that

$$\sup_n|\bar{x}^n|_T<\infty,\ \sup_n|\bar{u}^n|_T<\infty.$$

Further note that x^n and u^n are Lipschitz continuous with Lipschitz constants only depending on T. Using Arzela-Ascoli Theorem, there exists a uniformly convergent subsequence of $(\bar{x}^n,\bar{u}^n)$, and denote the subsequence by $(\bar{x}^{n_k},\bar{u}^{n_k})$ and its limit by $(\bar{x},\bar{u})$. It is easy to see that $\bar{u}$ is an admissible control for the FCP associated with x_0 and $\bar{\mathcal{M}}$, and so (7.5) follows. ∎

Proof of Theorem 4.8. It follows from Proposition 7.3 and Lemma 7.4. ∎

Proof of Theorem 4.9. From Theorem 4.8 and Proposition 7.1,

$$\bar{\mathcal{J}}^n(U^{n,*};X^n(0),\mathcal{M}^n)\to\bar{\mathcal{J}}^*(\bar{x}(0),\bar{\mathcal{M}}).$$

So it suffices to show that

$$\frac{1}{n}\bar{\mathcal{J}}(\mathbb{E}(U^{n,*});\mathbb{E}(X^n(0)),\mathcal{M}^n)\to\bar{\mathcal{J}}^*(\bar{x}(0),\bar{\mathcal{M}}).$$

From the proof of Proposition 4.4, it suffices to show that

$$|\mathbb{E}(U^{n,*})-u^*|_T\to 0. \tag{7.6}$$

Note that (7.6) follows from Proposition 7.3, Lemmas 7.4, and Proposition 7.1. The result follows. ∎

Proof of Proposition 5.1. Using the arguments in the proof of Proposition 4.4, it can be shown that $\limsup_{N\to\infty}\bar{\mathcal{J}}(\mathbb{E}(X(0)),\hat{\mathcal{M}}^N)\le\bar{\mathcal{J}}^*(\mathbb{E}(X(0)),\mathcal{M})$.

For large enough N, the shadow price k^N satisfies the linear system:

$$\dot{k}^N(t)=-\mathbf{1}+(\hat{d}^N+\hat{\varrho}^N-(\hat{\rho}^N)')k^N(t),t\in[0,T],\ k^N(T)=\hat{\beta}^N.$$

From Proposition 3.3, the optimal allocation rate

$$r_h^N(t)\in\arg\max_{z\in\mathbb{R}_+^I}\left\{(\alpha_h^N-k^N(t))'z:\mathbf{1}'z\le\mu_h^N(t)\right\},\ h\in\mathbb{H}.$$

A priority list $Z^{h,N}(t)=(Z_{\langle 1\rangle}^{h,N}(t),\dots,Z_{\langle I\rangle}^{h,N}(t))'$ at time t for organs of type h can be constructed based on the components of $\alpha_h^N-k^N(t)$ following the way below (4.4).

Let $\mathcal{H}_h(t)$ denote the set of the indices of the (tie) largest components of $\alpha_h^N - k^N(t)$, and define

$$\tilde{Z}^{h,N}_{\langle 1\rangle}(t) = \begin{cases} Z^h_{\langle 1\rangle}(t), & \text{if } Z^h_{\langle 1\rangle}(t) \in \mathcal{H}_h(t), \\ Z^{h,N}_{\langle 1\rangle}(t), & \text{otherwise.} \end{cases}$$

Define the control associated with $\tilde{Z}^{h,N}$ as follows:

$$\tilde{u}^{N,*}_{hi}(t) = \int_0^t \mu^N_h(s)\mathbf{1}(\tilde{Z}^{h,N}_{\langle 1\rangle}(s) = i)ds.$$

It is clear that $\tilde{u}^{N,*}$ is an optimal control for the FCP associated with $\mathbb{E}(X(0))$ and $\hat{\mathcal{M}}^N$. In the following we show that $|\tilde{Z}^{h,N} - Z^h|_T \to 0$ as $N \to \infty$. From the construction of both $\tilde{Z}^{h,N}$ and Z^h, it suffices to show that $|k^N - k|_T \to 0$. Note that

$$\begin{aligned} \sup_N \sup_{0\le t\le T} |k^N(t)| &\le \sup_N \int_0^T |-\mathbf{1} + (\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)')k^N(t)|dt \\ &\le IT + \sup_N |\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)'| \int_0^T |k^N(t)|dt. \end{aligned}$$

By Gronwall's inequality, we have that

$$\sup_N \sup_{0\le t\le T} |k^N(t)| \le IT \exp\{\sup_N |\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)'|\} < \infty.$$

Next the uniform continuity of $k(t)$ on $[0,T]$ implies that

$$\sup_{0\le t\le T} |k(t)| < \infty.$$

It follows that

$$\begin{aligned} \sup_{0\le t\le T} |k^N(t) - k(t)| &\le \int_0^T |(\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)')k^N(t) - (d + \varrho - \rho')k(t)|dt \\ &\le \int_0^T |(\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)')k^N(t) - (d + \varrho - \rho')k^N(t)|dt \\ &\quad + \int_0^T |(d + \varrho - \rho')k^N(t) - (d + \varrho - \rho')k(t)|dt \\ &\le \sup_N \sup_{0\le t\le T} |k^N(t)||(\hat{d}^N + \hat{\varrho}^N - (\hat{\rho}^N)') - (d + \varrho - \rho')|T \\ &\quad + |d + \varrho - \rho'| \int_0^T \sup_{0\le t\le \tau} |k^N(t) - k(t)|d\tau. \end{aligned}$$

Using Gronwall's inequality again yields that $\sup_{0\le t\le T} \left|k^N(t) - k(t)\right| \to 0$.

At last, we note that if $\alpha_h - k(t)$ has a unique largest component for almost everywhere $t \in [0,T]$, then $\tilde{Z}^N(t) = Z^N(t)$ for almost everywhere $t \in [0,T]$ when N is large enough. It follows now that $|Z^{h,N} - Z^h|_T \to 0$, and so $|u^{N,*} - u^*|_T \to 0$. ◻

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