On classes of Bitcoin-inspired infinite-server queueing systems

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Abstract

We present a 'Palm calculus' approach for deriving the stationary distribution of the G/M/∞-like Bitcoin queueing model recently studied in the work of Frolkova and Mandjes [11]. We then build on this technique by further showing that a similar approach can be used to study the time-dependent behavior of Markovian variants of this queueing system, at a considerable level of generality. Next, we further explain how an analogous M_t/G/∞-like Bitcoin queueing model can be analyzed as well, through an entirely different technique that also makes use of Palm distributions and point process techniques.

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1 Introduction

In the recent work of Frolkova and Mandjes [11], the authors study an infinite-server queueing system, where customers arrive in accordance to a renewal process having nonarithmetic (i.e. nonlattice) interrenewal distributions. The servers within this system are labeled as server 1, server 2, server 3, etc., and each server processes work at unit rate. Each arriving customer brings an exponentially distributed amount of work—having rate μ—for processing, and an arriving customer always joins the lowest-numbered idle server.

Unlike the classical infinite-server queue, in this queueing system jobs interact. Namely, as soon as a customer at server i has all of its work processed, it, along with all customers undergoing processing at lower-numbered servers, immediately leave the system. As soon as this batch departure occurs, any customer present at buffer $i + k$ immediately before the batch departure moves to buffer k, $k \geq 1$, where processing is resumed. This type of customer interaction is referred to in [11] as FIFO-batch departures, where FIFO stands for 'First-In-First-Out'. The authors of [11] designed this simple queueing system in order to better understand the effect of possible discrepancies among blockchain versions possessed by different users within the Bitcoin network, due to communication delays between users: this is important, as such discrepancies take away from the overall 'trustworthiness' of the entire system. Readers looking for more information on Bitcoin are referred to [11] and the papers cited therein.

Using the notation scheme of [11], for each real number $t \geq 0$, let $Q(t)$ denote the total number of customers present in the system at time t. In [11], the authors argue that because services are exponentially distributed, the distribution of the entire queue-length process $\{Q(t); t \geq 0\}$ remains the same if customers instead depart under the LIFO-batch departure rule, where LIFO stands for 'Last-In-First-Out'. Under this rule, as soon as a customer at server i completes processing, it, along with all customers undergoing processing at higher-labeled servers, immediately leave the system. This is shown in [11] to be an extremely useful observation: indeed, the authors elegantly show in Lemma 1 of [11] that the LIFO-batch departure scheme clearly explains why the busy period of this queueing system is exponentially distributed with rate μ, meaning that the busy period distribution does not depend on the interarrival distribution, or even on the arrival rate. The authors further use the LIFO-batch departure rule in Lemma 2 of [11] to derive the stationary distribution of $\{Q(t); t \geq 0\}$, and when the arrival process is further assumed to be Poisson, Lemma 5 of [11] shows that a recursive expression for the probability mass function of $Q(t)$ can be derived for each $t \geq 0$. Lemma 3 of [11] gives a recursion for the moments of the stationary distribution when arrivals are further assumed to be Poisson, and Lemma 4 of [11] shows that under a fluid-scaling regime, the stationary distribution converges weakly to a Weibull distribution. The remainder of [11] is devoted to their main result—Theorem 1—which shows that under a fluid scaling regime, the entire queue-length process converges weakly to a growth-collapse model.

Our first goal is to further emphasize just how valuable the LIFO-batch departure scheme from [11] is with regards to computing quantities associated with their Bitcoin model, as well as other population/queueing models where organisms/customers depart in a manner similar to customers in their Bitcoin model: examples of such population models include those studied in Brockwell et al [7] that feature uniform catastrophes. Our second goal is to show that the more realistic FIFO-batch departure discipline can be used directly to analyze a natural $\mathrm{M}_t/\mathrm{G}/\infty$-like Bitcoin queue variant of the $\mathrm{G}/\mathrm{M}/\infty$-like Bitcoin queue analyzed in [11].

This paper is organized as follows. In Section 2, we use the Palm measure induced by the stationary version of the renewal arrival process to provide an alternative derivation of the stationary distribution of the Bitcoin queueing model introduced in [11]. Our approach further yields the stationary distribution of the number of customers observed in the system by an arrival. We also quickly sketch a similar rate-conservation argument based on the same key ideas, that does not make explicit use of stationary Palm theory. Next, in Section 3, we consider a purely Markovian variant of this infinite-server queueing model, where arrivals are allowed to occur in batches, and where the arrival rates and batch size distributions depend on the number of customers present in the system. We show that in general, the Laplace transforms of the transition functions can be computed recursively, and under additional conditions, the transition functions themselves can be computed recursively, or even expressed in closed-form. Finally, in Section 4, we analyze a $\mathrm{M}_t/\mathrm{G}/\infty$-like Bitcoin queueing model operating under the FIFO-batch departures discipline. There it will be shown that the probability mass function of the number of customers in the system can be expressed explicitly, by making use of different techniques from the theory of point processes. Not only is this model interesting because it allows us to incorporate non-stationarity into the model, it is mathematically interesting because in order to analyze this model, we make explicit use of the FIFO-batch departures discipline.

2 The Case of Renewal Arrivals

Consider a simple, stationary renewal (counting) process $A := \{A(s,t];\ s,t \in \mathbb{R},\ s < t\}$ having as its points $\{T_n\}_{n\in\mathbb{Z}}$, where $\ldots < T_{-2} < T_{-1} < T_0 \le 0 < T_1 < T_2 < \ldots$. The connection between A and its points is given by

$$A(s,t] := \sum_{n\in\mathbb{Z}} \mathbf{1}(T_n \in (s,t]), \quad s,t \in \mathbb{R},\ s < t$$

which is a mathematical way of saying $A(s,t]$ counts the number of points found in the interval $(s,t]$, since $\mathbf{1}(\cdot)$ represents an indicator function, equal to 1 if the statement $(\cdot)$ is true, and 0 if the statement is false. Each point T_n has associated with it a mark σ_n, and we assume throughout that $\{\sigma_n\}_{n\in\mathbb{Z}}$ constitutes an i.i.d. sequence of exponential random variables with rate μ, independent of A.

We can use both the points $\{T_n\}_{n\in\mathbb{Z}}$ and their exponential marks $\{\sigma_n\}_{n\in\mathbb{Z}}$ to construct a stationary version of the Bitcoin queueing model of [11]. Indeed, each T_n represents the arrival time of a customer to a queueing system that consists of infinitely many servers, and σ_n represents the amount of work this arrival brings to the system for processing. From these arrival points and their exponential marks, we can construct a queueing process $\{Q(t); t \in \mathbb{R}\}$, where $Q(t)$ represents the number of customers present in the system at time $t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, we can express $Q(t)$ in the following way:

$$Q(t) = \sum_{n\in\mathbb{Z}} \mathbf{1}(T_n \le t)\mathbf{1}(T_n + W_n > t)$$

where

$$W_n := \inf_{m\in A_n} \{\sigma_m - (T_n - T_m)\}, \quad A_n := \{m \in \mathbb{Z} : m \le n,\ T_m + W_m > T_n\}.$$

In words, A_n represents a set which contains the integer n, as well as the arrival indices corresponding to customers found in the system at the moment the customer arriving at time T_n arrives, and W_n is the minimum of σ_n, as well as the remaining amounts of unprocessed work discovered by the customer upon arrival at time T_n. What makes this system different from the infinite-server queue is the interaction that occurs between the customers in the system. If we were trying to model the infinite-server queue where customers do not interact with one another, we would disregard usage of both A_n and W_n, and simply replace each W_n found in the definition of $Q(t)$ with σ_n.

Some readers may note that we do not give a completely rigorous construction of the stationary Bitcoin model, i.e. we do not construct it as a shift-invariant functional of a well-defined stationary flow, but one can easily do so by first constructing a stationary $G/M/\infty$ queue on $\mathbb{R}$, as is done rigorously in Chapter 2 of [3]. Once this construction has been made, the Bitcoin model can be considered as a functional of the constructed $G/M/\infty$ queue: more specifically, the $G/M/\infty$ queue can be used to give a more rigorous description of the A_n sets used in the Bitcoin model, by making use of the fact that the $G/M/\infty$ queue empties infinitely often over $(-\infty, t]$ for each $t \in \mathbb{R}$, and when the $G/M/\infty$ queue is empty, the constructed Bitcoin model must be empty as well. This observation forces each of the sets within the collection $\{A_n\}_{n\in\mathbb{Z}}$ to be finite, and it also ensures that for each $t \in \mathbb{R}$, there exists a largest integer $n_0(t) \le t$ satisfying $A_{n_0(t)} = \{n_0(t)\}$. This observation makes our definition of $Q(t)$ rigorous for each $t \in \mathbb{R}$, as it allows us to construct $Q(t)$ simply by looking from time $T_{n_0(t)}$ onward.

Our first objective is to present a new derivation of the distribution of $Q(0)$ under our underlying probability measure $\mathbb{P}$. By stationarity, the distribution of $Q(0)$ is also the distribution of $Q(t)$ for each $t \in \mathbb{R}$, and this distribution can be interpreted as the limiting distribution of the Bitcoin queueing model, at least in the case where the interrenewals are nonarithmetic.

The derivation we present of the law of $Q(0)$ under $\mathbb{P}$ requires us to simultaneously study the law of $Q(0-) := \lim_{s \uparrow 0} Q(s)$ under the Palm measure $\mathcal{P}_0$ induced by the stationary arrival process A. Readers not familiar with stationary Palm measures are recommended to consult Chapter 1 of Baccelli and Brémaud [3]: intuitively, $\mathcal{P}_0(C)$ can be interpreted as the probability of the event C, conditional on $T_0 = 0$. Hence, the law of $Q(0-)$ under $\mathcal{P}_0$ represents the distribution of the number of customers in the system immediately before an arrival occurs. The famous PASTA property—see Brémaud [5] for the Palm version of PASTA, as well as Wolff [14] for the classical long-run average interpretation of PASTA—shows that, under Poisson arrivals, the law of $Q(0)$ under $\mathbb{P}$ is the same as the law of $Q(0-)$ under $\mathcal{P}_0$, but in most cases these two laws are not equal.

Theorem 2.1 *The law of $Q(0-)$ under the Palm measure $\mathcal{P}_0$ induced by A satisfies*

$$\mathcal{P}_0(Q(0-) \geq k) = \prod_{\ell=1}^{k} \mathcal{E}_0[e^{-\ell \mu T_1}], \qquad k = 1, 2, 3, \dots. \tag{1}$$

Furthermore, the law of $Q(0)$ under $\mathbb{P}$ is given by

$$\mathbb{P}(Q(0) \geq k) = \frac{\lambda}{k\mu}(1 - \mathcal{E}_0[e^{-k\mu T_1}]) \prod_{\ell=1}^{k-1} \mathcal{E}_0[e^{-\ell \mu T_1}], \qquad k = 1, 2, 3, \dots \tag{2}$$

where $\lambda := 1/\mathcal{E}_0[T_1]$.

Formula (2) is equivalent to the expression found in Lemma 2 of [11], since the law of T_1 under $\mathcal{P}_0$ is just the law of a typical interrenewal interval.

Proof First, observe that the stationary process $\{Q(t); t \in \mathbb{R}\}$ must reach state zero infinitely many times in $(-\infty, 0]$. This implies that for each integer $k \geq 1$,

$$\mathbf{1}(Q(0) \geq k) \quad = \quad \int_{-\infty}^{0} \mathbf{1}(Q(s-) = k-1)\mathbf{1}(\tau_{k-1}(s) > 0)A(ds)$$

where $\tau_{k-1}(s) := \inf\{t \geq s : Q(t) \leq k-1\}$. Taking the expectation of both sides, while further applying the stationary version of Campbell's formula on the right-hand-side (see page 18 of [3]) gives

$$\mathbb{P}(Q(0) \geq k) = \lambda \mathcal{P}_0(Q(0-) = k-1)\mathcal{E}_0[\tau_{k-1}(0) \mid Q(0-) = k-1] = \frac{\lambda}{k\mu}\mathcal{P}_0(Q(0-) = k-1) \tag{3}$$

where $\lambda := 1/\mathcal{E}_0[T_1]$ is the arrival rate of customers to the system, which is simply 1 divided by the expected length of a typical interrenewal interval. Readers should note that Formula (3) follows from the observation

$$\mathcal{E}_0[\tau_{k-1}(0) \mid Q(0-) = k-1] = \frac{1}{k\mu}$$

since, conditional on $Q(0-)=k-1$ under $\mathcal{P}_0$, at time 0 exactly k customers are present in the system, and the queue-length process reaches a level at or below $k-1$ as soon as one of these customers leave the system: this happens at a time that corresponds to a minimum of k i.i.d. exponentially distributed random variables with rate μ.

A simple rewrite of (3) further shows that for each integer $k \geq 1$,

$$\mathcal{P}_0(Q(0-) \geq k-1) - \mathcal{P}_0(Q(0-) \geq k) = \frac{k\mu}{\lambda}\mathbb{P}(Q(0) \geq k). \tag{4}$$

The next step consists of using the Inversion formula—see pg. 20 of [3]—to get another recursion that relates the law of $Q(0)$ under $\mathbb{P}$ to the law of $Q(0-)$ under $\mathcal{P}_0$. Indeed, for each integer $k \geq 1$, we have under $\mathcal{P}_0$ that (again, recall that under $\mathcal{P}_0$, $T_0 = 0$ with probability one)

$$\int_0^{T_1} \mathbf{1}(Q(s) \geq k)ds = \mathbf{1}(Q(0-) \geq k-1)\min(T_1, e_{k\mu})$$

with probability one, where $e_{k\mu}$ is an exponentially distributed random variable with rate $k\mu$, and independent of T_1 under $\mathcal{P}_0$. Applying now the Inversion formula, we find that for each integer $k \geq 1$,

$$\begin{aligned} \mathbb{P}(Q(0) \geq k) &= \lambda\mathcal{E}_0\left[\int_0^{T_1} \mathbf{1}(Q(s) \geq k)ds\right] \\ &= \lambda\mathcal{E}_0\left[\mathbf{1}(Q(0-) \geq k-1)\min(T_1, e_{k\mu})\right] \\ &= \lambda\mathcal{P}_0(Q(0-) \geq k-1)\mathcal{E}_0[\min(T_1, e_{k\mu})] \end{aligned} \tag{5}$$

where the last equality follows from $\mathbf{1}(Q(0-) \geq k-1)$ being independent of both T_1 and $e_{k\mu}$ under the measure $\mathcal{P}_0$. Next, observe that since

$$\mathcal{E}_0[\min(T_1, e_{k\mu})] = \int_0^\infty \mathcal{P}_0(T_1 > t)e^{-k\mu t}dt = \mathcal{E}_0\left[\int_0^{T_1} e^{-k\mu t}dt\right] = \frac{1-\mathcal{E}_0[e^{-k\mu T_1}]}{k\mu}$$

we may claim that for each integer $k \geq 1$,

$$\mathbb{P}(Q(0) \geq k) = \frac{\lambda(1-\mathcal{E}_0[e^{-k\mu T_1}])}{k\mu}\mathcal{P}_0(Q(0-) \geq k-1). \tag{6}$$

Formulas (4) and (6) can be combined to find the law of $Q(0-)$ under $\mathcal{P}_0$, as well as the law of $Q(0)$ under $\mathbb{P}$. Indeed, applying (6) to the right-hand-side of (4) gives

$$\mathcal{P}_0(Q(0-) \geq k-1) - \mathcal{P}_0(Q(0-) \geq k) = (1-\mathcal{E}_0[e^{-k\mu T_1}])\mathcal{P}_0(Q(0-) \geq k-1)$$

or, equivalently, for each integer $k \geq 1$,

$$\mathcal{P}_0(Q(0-) \geq k) = \mathcal{E}_0[e^{-k\mu T_1}]\mathcal{P}_0(Q(0-) \geq k-1).$$

This gives

$$\mathcal{P}_0(Q(0-) \geq k) = \prod_{\ell=1}^{k} \mathcal{E}_0[e^{-\ell\mu T_1}]$$

which proves (1). To show (2), use (1) within (3): for each integer $k \geq 1$,

$$\mathbb{P}(Q(0) \geq k) = \frac{\lambda}{k\mu}\mathcal{P}_0(Q(0-) = k-1) = \frac{\lambda(1 - \mathcal{E}_0[e^{-k\mu T_1}])}{k\mu}\prod_{\ell=1}^{k-1}\mathcal{E}_0[e^{-\ell\mu T_1}].$$

This completes the proof. □

Remark It is possible to present a more elementary, and more intuitive, version of our derivation of Theorem 2.1 that does not invoke the use of Palm measures and the construction of stationary processes on the doubly-infinite time axis $\mathbb{R}$. Here is a brief sketch of the proof.

We begin by justifying the derivation of (4) outside of a Palm measure context. Observe that Formula (3) can alternatively be expressed as

$$k\mu\mathbb{P}(Q(0) \geq k) = \lambda\mathcal{P}_0(Q(0-) = k-1). \tag{7}$$

To verify this equality without Palm measures—please accept for the moment our usage of $\mathcal{P}_0$, as this will be addressed soon—note that the left-hand-side of (7) represents the long-run rate at which transitions are made from the set $\{k, k+1, k+2, \ldots\}$ to $\{0, 1, 2, \ldots, k-1\}$, while the right-hand-side of (7) represents the long-run rate at which transitions are made from $\{0, 1, 2, \ldots, k-1\}$ to $\{k, k+1, k+2, \ldots\}$, and these two long-run averages must agree. The underlying measure $\mathbb{P}$ can be used on the left-hand-side, since while the process is in $\{k, k+1, k+2, \ldots\}$, transitions to $\{0, 1, 2, \ldots, k-1\}$ can be said to occur among a subset of points generated in accordance to a Poisson process with rate $k\mu$, and this allows us to invoke the classical long-run-average version of the PASTA property, as in [14]. Furthermore, $\mathcal{P}_0(Q(0-) = k-1)$ should be interpreted as the long-run fraction of customers that observe $k-1$ other customers in the system upon arrival (we use $\mathcal{P}_0$ here, as this also happens to be a Palm probability). Readers may notice that this is a rate-conservation argument, and is highly analogous to the argument used in Section X.5 of Asmussen [2] within the context of $G/M/1$ queues.

It remains to similarly justify (6). Given arrivals occur in accordance to a renewal process and each customer brings exponentially distributed amounts of work, Formula (5) can also be rigorously stated using only semi-regenerative theory, where all quantities can be interpreted as limits: see Proposition 5.2 on page 212 of Asmussen [2].

3 The Markovian Case

Suppose now that customers arrive to the queueing system from two independent, state-dependent Poisson sources, which we refer to throughout this section as Source 1 and Source 2. Source 1 is a state-dependent Poisson process, meaning while there are n customers in the system, new customers arrive one-at-a-time from Source 1 in accordance to a Poisson process with rate λ_n. It is convenient to model Source 1 as a collection of independent, homogeneous Poisson processes $\{A^{(1)}_{n,n+1}\}_{n\geq 0}$, where $A^{(1)}_{n,n+1}$ is a homogeneous Poisson process with rate λ_n.

Source 2 is a state-dependent Poisson process with batch arrivals, meaning while there are n customers in the system, new batches of customers arrive in accordance to a Poisson process with rate γ_n, where the arriving batch contains exactly k customers with probability $g_{n,n+k}$, $k \geq 1$. As with Source 1, it helps to model Source 2 with a collection of homogeneous Poisson processes

$\{A^{(2)}_{n,n+k}\}_{n\geq 0,k\geq 1}$, where $A^{(2)}_{n,n+k}$ is a homogeneous Poisson process with rate $\gamma_n g_{n,n+k}$. We assume throughout that for each integer $n \geq 0$,

$$\sum_{k=1}^{\infty} g_{n,n+k} = 1.$$

Finally, departures from the queueing system can be modeled with a collection of Poisson processes $\{D_m\}_{m\geq 1}$, where D_m is a homogeneous Poisson process with rate μ. The process D_m governs how services take place at server m, for each integer $m \geq 1$.

These homogeneous Poisson processes can be used to build $\{Q(t); t \geq 0\}$ in the following way. While $Q(t)$ is in state n, the Poisson processes $A^{(1)}_{n,n+1}$, $\{A^{(2)}_{n,n+k}\}_{k\geq 1}$, and $\{D_m\}_{1\leq m\leq n}$ are all said to be active. This means that if a point from one of these processes occurs at time t, and $Q(t-) = n$ (where $Q(t-)$ is the left-hand-limit of Q at time t), then that point governs the transition of Q occurring at time t. For instance, if at time t, $Q(t-) = n$, and $A^{(1)}_{n,n+1}$ has a point at time t, then $Q(t) = n + 1$. Similarly, if $Q(t-) = 5$ and D_2 has a point at time t, then $Q(t) = 1$, since the customer at server 2 completed processing at time t. We can explicitly express $Q(t)$ in terms of these Poisson processes using 'Poisson integrals': readers interested in seeing the details of how to carry out such a construction are referred to Chapter 9 of Brémaud [6].

Under these new arrival processes, $\{Q(t); t \geq 0\}$ is a CTMC having state space $E = \{0, 1, 2, 3, \ldots\}$, whose transition rate matrix $\mathbf{Q}$ has off-diagonal elements that satisfy

$$q(n, n + k) \quad = \quad \lambda_n \mathbf{1}(k = 1) + \gamma_n g_{n,n+k}, \qquad n \geq 0, \;\; k \geq 1$$

and

$$q(n, j) \quad = \quad \mu, \qquad n \geq 1, \;\; 0 \leq j \leq n - 1.$$

All other off-diagonal entries of $\mathbf{Q}$ are equal to zero.

Readers should note that this CTMC generalizes the uniform-catastrophe model featured in Brockwell et al [7]: their interest in population models most likely led them to consider only the case where $\lambda_n = n\lambda$, and $\gamma_n = \gamma$ for each integer $n \geq 0$.

Further associated with the CTMC $\{Q(t); t \geq 0\}$ is the set of transition functions $\{p_k\}_{k\geq 0}$, where for each integer $k \geq 0$,

$$p_k(t) := \mathbb{P}(Q(t) = k), \qquad t \geq 0.$$

We suppress any explicit reference to the initial distribution, as this can be arbitrarily chosen. Each transition function p_k has a Laplace transform π_k that is well-defined on the open half-plane $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$ as

$$\pi_k(\alpha) := \int_0^{\infty} e^{-\alpha t} p_k(t) dt, \qquad \alpha \in \mathbb{C}_+.$$

The following Theorem shows that the Laplace transforms π_k, $k \geq 0$, can be computed recursively. Our method for computing these transforms requires us to define the quantities $\overline{f}_{j,k}$, where for each integer $j \geq 0$, and each integer $k \geq j + 1$,

$$\overline{f}_{j,k} = \sum_{\ell=k-j}^{\infty} g_{j,j+\ell}.$$

Theorem 3.1 *For each $\alpha \in \mathbb{C}_+$,*

$$\pi_0(\alpha) = \frac{\mu + \alpha\mathbb{P}(Q(0) = 0)}{\alpha(\mu + \alpha + \lambda_0 + \gamma_0)} \tag{8}$$

and for each integer $k \geq 1$,

$$\pi_k(\alpha) = \left[\frac{1}{\alpha} - \frac{\mathbb{P}(Q(0) \geq k+1)}{(k+1)\mu + \alpha} - \sum_{j=0}^{k-1}\left[1 + \frac{\gamma_j \overline{f}_{j,k+1}}{(k+1)\mu + \alpha}\right]\pi_j(\alpha)\right]\left[1 + \frac{\lambda_k + \gamma_k}{(k+1)\mu + \alpha}\right]^{-1}. \tag{9}$$

This result, in conjunction with the transform inversion algorithm of e.g. Abate and Whitt [1] or den Iseger [12], can be used to numerically compute each transition function of the CTMC.

Proof For each real number $s \geq 0$, define for each integer $k \geq 0$ the random variable

$$\tau_k(s) := \inf\{u \geq s : Q(u) \leq k\}.$$

First observe that for each integer $k \geq 0$,

$$\begin{aligned}\mathbf{1}(Q(t) \geq k+1) &= \mathbf{1}(Q(0) \geq k+1)\mathbf{1}(\tau_k(0) > t) + \int_0^t \mathbf{1}(Q(s-) = k)\mathbf{1}(\tau_k(s) > t)A^{(1)}_{k,k+1}(ds) \\ &+ \sum_{j=0}^{k}\sum_{\ell=k+1-j}^{\infty}\int_0^t \mathbf{1}(Q(s-) = j)\mathbf{1}(\tau_k(s) > t)A^{(2)}_{j,j+\ell}(ds).\end{aligned}$$

The right-hand-side of this equality is easy to understand. In order for $Q(t)$ to be greater than or equal to $k+1$, either (1) the process starts at a level at or above $k+1$, and never moves below $k+1$ until after time t, or (2) at some point before time t, the process moves from a state in $\{0, 1, 2, \ldots, k\}$ to a state in $\{k+1, k+2, \ldots\}$, and stays in this set until after time t.

Proceeding as in [9], taking expectations of both sides, while further applying both the Campbell-Mecke formula and the time-dependent PASTA property, further yields

$$\begin{aligned}\mathbb{P}(Q(t) \geq k+1) &= \mathbb{P}(Q(0) \geq k+1)e^{-(k+1)\mu t} + \lambda_k \int_0^t \mathbb{P}(Q(s) = k)e^{-(k+1)\mu(t-s)}ds \\ &+ \sum_{j=0}^{k}\gamma_j \overline{f}_{j,k+1}\int_0^t \mathbb{P}(Q(s) = j)e^{-(k+1)\mu(t-s)}ds. \end{aligned} \tag{10}$$

This derivation of (10) is a time-dependent version of the argument used to derive Formula (3) in Section 2. In the case where $Q(0) = 0$ with probability one, (10) could also be derived from the formula found at the top of page 124 of Latouche and Ramaswami [13].

The key observation used in this derivation of Formula (10) is the following: as soon as the process makes a transition from a state $j \leq k$ to a state $j + \ell \geq k+1$ at some time $s < t$, the distribution of the amount of time it takes Q to reach the set $\{0, 1, 2, \ldots, k\}$ is exponential with rate $(k+1)\mu$, since the only way this can happen is if one of the customers at buffers $1, 2, \ldots, k, k+1$ departs before time t. This is a simple property associated with the LIFO-batch departures service discipline.

The next step is to take the Laplace transform of both sides of Formula (10). Doing this shows that for each $\alpha \in \mathbb{C}_+$,

$$\sum_{\ell=k+1}^{\infty} \pi_\ell(\alpha) = \frac{\mathbb{P}(Q(0) \geq k+1)}{(k+1)\mu + \alpha} + \frac{\lambda_k}{(k+1)\mu + \alpha}\pi_k(\alpha) + \sum_{j=0}^{k} \frac{\gamma_j \overline{f}_{j,k+1}}{(k+1)\mu + \alpha}\pi_j(\alpha) \tag{11}$$

or, equivalently,

$$\frac{1}{\alpha} - \sum_{\ell=0}^{k} \pi_\ell(\alpha) = \frac{\mathbb{P}(Q(0) \geq k+1)}{(k+1)\mu + \alpha} + \frac{\lambda_k}{(k+1)\mu + \alpha}\pi_k(\alpha) + \sum_{j=0}^{k} \frac{\gamma_j \overline{f}_{j,k+1}}{(k+1)\mu + \alpha}\pi_j(\alpha) \tag{12}$$

Solving for $\pi_k(\alpha)$ in (12) then yields (9), while setting $k = 0$ in (12) and solving further yields (8). □

It is worth observing that the Laplace transform $\pi_0(\alpha)$ can be inverted easily: for each $t \geq 0$,

$$p_0(t) = \frac{\mu}{\lambda_0 + \gamma_0 + \mu}(1 - e^{-(\lambda_0+\gamma_0+\mu)t}) + \mathbb{P}(Q(0) = 0)e^{-(\lambda_0+\gamma_0+\mu)t}.$$

3.1 The case where $\lambda_k = \lambda$, $\gamma_k = \gamma$

The form of the transition functions simplify whenever $\lambda_k := \lambda$ and $\gamma_k := \gamma$ for each integer $k \geq 0$.

Theorem 3.2 *For each integer $k \geq 0$, we have*

$$\mathbb{P}(Q(t) = k) = C_{k,0} + \sum_{\ell=1}^{k+1} C_{k,\ell} e^{-(\lambda+\gamma+\ell\mu)t}$$

where the $\{C_{k,\ell}\}_{k\geq 0, 0\leq \ell \leq k+1}$ terms satisfy a recursive structure. More specifically,

$$C_{0,0} = \frac{\mu}{\lambda + \gamma + \mu}, \qquad C_{0,1} = \mathbb{P}(Q(0) = 0) - \frac{\mu}{\lambda + \gamma + \mu}$$

and for each integer $k \geq 0$,

$$C_{k+1,0} = \frac{(k+2)\mu}{\lambda + \gamma + (k+2)\mu}\left[1 - \sum_{j=0}^{k} C_{j,0}\right] - \frac{1}{\lambda + \gamma + (k+2)\mu}\sum_{j=0}^{k} \gamma \overline{g}_{j,k+2} C_{j,0},$$

$$C_{k+1,\ell} = \frac{\lambda + \gamma - (k+2-\ell)\mu}{(k+2-\ell)\mu} \sum_{j=\ell-1}^{k} C_{j,\ell} - \frac{1}{(k+2-\ell)\mu} \sum_{j=\ell-1}^{k} \gamma \overline{g}_{j,k+2} C_{j,\ell}$$

for each $\ell \in \{1, 2, \ldots, k+1\}$, and

$$C_{k+1,k+2} = \mathbb{P}(Q(0) = k+1) - \sum_{\ell=0}^{k+1} C_{k+1,\ell}.$$

Proof Formula (10) from the proof of Theorem 3.1 can alternatively be stated as follows: for each integer $k \geq 0$, and each real $t \geq 0$,

$$1 - \sum_{j=0}^{k} p_j(t) = \mathbb{P}(Q(0) \geq k+1)e^{-(k+1)\mu t} + \lambda \int_0^t p_k(s)e^{-(k+1)\mu(t-s)}ds + \sum_{j=0}^{k} \gamma \overline{f}_{j,k+1} \int_0^t p_j(s)e^{-(k+1)\mu(t-s)}ds.$$

We have already established that for each real number $t \geq 0$,

$$p_0(t) = C_{0,0} + C_{0,1}e^{-(\lambda+\gamma+\mu)t}$$

where

$$C_{0,0} = \frac{\mu}{\lambda+\gamma+\mu}, \qquad C_{0,1} = \mathbb{P}(Q(0) = 0) - \frac{\mu}{\lambda+\gamma+\mu}.$$

From here we use induction. Suppose we have determined that for $0 \leq j \leq k$,

$$p_j(t) = C_{j,0} + \sum_{\ell=1}^{j+1} C_{j,\ell}e^{-(\lambda+\gamma+\ell\mu)t}, \qquad t \geq 0.$$

To find $p_{k+1}(t)$, observe from Formula (10)—after replacing k with $k+1$—that p_{k+1} satisfies the integral equation

$$p_{k+1}(t) + (\lambda+\gamma)\int_0^t p_{k+1}(s)e^{-(k+2)\mu(t-s)}ds = G(t) \tag{13}$$

where

$$G(t) := 1 - \sum_{j=0}^{k} p_j(t) - \mathbb{P}(Q(0) \geq k+2)e^{-(k+2)\mu t} - \sum_{j=0}^{k} \gamma_j \overline{g}_{j,k+2} \int_0^t p_j(s)e^{-(k+2)\mu(t-s)}ds. \tag{14}$$

To solve (13), first multiply both sides of (13) by $e^{(k+2)\mu t}$ and define $f_{k+1}(t) := p_{k+1}(t)e^{(k+2)\mu t}$. Doing so gives

$$f_{k+1}(t) + (\lambda+\gamma)\int_0^t f_{k+1}(s)ds = H(t) \tag{15}$$

where

$$H(t) = e^{(k+2)\mu t} - \sum_{j=0}^{k} e^{(k+2)\mu t}p_j(t) - \mathbb{P}(Q(0) \geq k+2) - \sum_{j=0}^{k} \gamma_j \overline{g}_{j,k+2} \int_0^t p_j(s)e^{(k+2)\mu s}ds. \tag{16}$$

Differentiating both sides of (15) shows that $f_{k+1}(t)$ also satisfies

$$f_{k+1}^{'}(t) + (\lambda+\gamma)f_{k+1}(t) = h(t) \tag{17}$$

where by the induction hypothesis,

$$
\begin{aligned}
h(t) &= (k+2)\mu e^{(k+2)\mu t} \\
&\quad - \sum_{j=0}^{k}\left[(k+2)\mu e^{(k+2)\mu t}\left[C_{j,0}+\sum_{\ell=1}^{j+1}C_{j,\ell}e^{-(\lambda+\gamma+\ell\mu)t}\right] - e^{(k+2)\mu t}\left[\sum_{\ell=1}^{j+1}(\lambda+\gamma+\ell\mu)C_{j,\ell}e^{-(\lambda+\gamma+\ell\mu)t}\right]\right] \\
&\quad - \sum_{j=0}^{k}\gamma_j\overline{g}_{j,k+2}e^{(k+2)\mu t}\left[C_{j,0}+\sum_{\ell=1}^{j+1}C_{j,\ell}e^{-(\lambda+\gamma+\ell\mu)t}\right] \\
&= e^{(k+2)\mu t}\left[D_{k+1,0}+\sum_{\ell=1}^{k+1}D_{k+1,\ell}e^{-(\lambda+\gamma+\ell\mu)t}\right] \qquad (18)
\end{aligned}
$$

with the coefficients $D_{k+1,0}, D_{k+1,1}, \ldots, D_{k+1,k+1}$ given by

$$D_{k+1,0} = (k+2)\mu\left[1-\sum_{j=0}^{k}C_{j,0}\right] - \sum_{j=0}^{k}\gamma\overline{g}_{j,k+2}C_{j,0}$$

and for $1 \le \ell \le k+1$,

$$D_{k+1,\ell} = (\lambda+\gamma-(k+2-\ell)\mu)\sum_{j=\ell-1}^{k}C_{j,\ell} - \sum_{j=\ell-1}^{k}\gamma\overline{g}_{j,k+2}C_{j,\ell}.$$

Solving (17) for $f_{k+1}(t)$ further yields

$$f_{k+1}(t) = \mathbb{P}(Q(0)=k+1)e^{-(\lambda+\gamma)t} + \int_0^t h(s)e^{-(\lambda+\gamma)(t-s)}ds$$

or

$$
\begin{aligned}
p_{k+1}(t) &= \mathbb{P}(Q(0)=k+1)e^{-(\lambda+\gamma+(k+2)\mu)t} + e^{-(\lambda+\gamma+(k+2)\mu)t}\int_0^t e^{(\lambda+\gamma)s}h(s)ds \\
&= \mathbb{P}(Q(0)=k+1)e^{-(\lambda+\gamma+(k+2)\mu)t} + e^{-(\lambda+\gamma+(k+2)\mu)t}\int_0^t\left[D_{k+1,0}e^{(\lambda+\gamma+(k+2)\mu)s}+\sum_{\ell=1}^{k+1}D_{k+1,\ell}e^{(k+2-\ell)\mu s}\right]ds \\
&= \mathbb{P}(Q(0)=k+1)e^{-(\lambda+\gamma+(k+2)\mu)t} + \frac{D_{k+1,0}}{\lambda+\gamma+(k+2)\mu}\left[1-e^{-(\lambda+\gamma+(k+2)\mu)t}\right] \\
&\quad + \sum_{\ell=1}^{k+1}\frac{D_{k+1,\ell}}{(k+2-\ell)\mu}\left[e^{-(\lambda+\gamma+\ell\mu)t}-e^{-(\lambda+\gamma+(k+2)\mu)t}\right] \\
&= C_{k+1,0}+\sum_{\ell=1}^{k+2}C_{k+1,\ell}e^{-(\lambda+\gamma+\ell\mu)t}
\end{aligned}
$$

where

$$C_{k+1,0} = \frac{D_{k+1,0}}{\lambda+\gamma+(k+2)\mu}$$

$$C_{k+1,k+2} = \mathbb{P}(Q(0) = k+1) - \left[\frac{D_{k+1,0}}{\lambda + \gamma + (k+2)\mu} + \sum_{\ell=1}^{k+1} \frac{D_{k+1,\ell}}{(k+2-\ell)\mu} \right]$$

and for $1 \le \ell \le k+1$,

$$C_{k+1,\ell} = \frac{D_{k+1,\ell}}{(k+2-\ell)\mu}.$$

This completes the derivation. □

3.2 The case where $\gamma_k = 0$ for $k \ge 0$

Much more can be said about the transition functions of $\{Q(t); t \ge 0\}$ if we restrict all arriving batches to be of size one. To model this restriction, it suffices to simply set $\gamma_k = 0$ for each $k \ge 0$, i.e. it suffices to only allow arrivals from Source 1. Our next result shows that, under this additional assumption, the transition functions can be expressed in closed-form. We assume for convenience that $\lambda_j + j\mu \neq \lambda_k + k\mu$ for any two distinct integers $j, k \ge 0$, but similar expressions can be computed if this assumption does not hold.

Theorem 3.3 *For each real number $t \ge 0$, we have*

$$\begin{aligned} \mathbb{P}(Q(t) \ge k) \quad &= \quad \sum_{\ell=1}^{k} \left[\prod_{j=1, j\neq\ell}^{k} \frac{\lambda_{j-1}}{(\lambda_{j-1} - \lambda_{\ell-1}) + (j-\ell)\mu} \right] \frac{\lambda_{\ell-1}}{\lambda_{\ell-1} + \ell\mu} (1 - e^{-(\lambda_{\ell-1} + \ell\mu)t}) \\ &\quad + \sum_{m=1}^{k} \frac{\mathbb{P}(Q(0) \ge m)}{\lambda_{m-1}} \sum_{\ell=m}^{k} \lambda_{\ell-1} \left[\prod_{j=m, j\neq\ell}^{k} \frac{\lambda_{j-1}}{(\lambda_{j-1} - \lambda_{\ell-1}) + (j-\ell)\mu} \right] e^{-(\lambda_{\ell-1} + \ell\mu)t}. \end{aligned}$$

Proof It is easier to establish this result with Laplace transforms. Starting with Formula (11), replace $k+1$ with k, and define $\Pi_k(\alpha) := \sum_{\ell \ge k} \pi_\ell(\alpha)$. Then (11) can be rewritten as

$$\Pi_k(\alpha) = \frac{\mathbb{P}(Q(0) \ge k)}{\lambda_{k-1} + k\mu + \alpha} + \frac{\lambda_{k-1}}{\lambda_{k-1} + k\mu + \alpha} \Pi_{k-1}(\alpha), \qquad k \ge 1$$

with $\Pi_0(\alpha) = 1/\alpha$. Next, an induction argument can be used to establish, for each integer $k \ge 1$,

$$\Pi_k(\alpha) = \frac{1}{\alpha} \prod_{\ell=1}^{k} \frac{\lambda_{\ell-1}}{\lambda_{\ell-1} + \ell\mu + \alpha} + \sum_{m=1}^{k} \frac{\mathbb{P}(Q(0) \ge m)}{\lambda_{m-1}} \prod_{\ell=m}^{k} \frac{\lambda_{\ell-1}}{\lambda_{\ell-1} + \ell\mu + \alpha}$$

It is not difficult to analytically invert $\Pi_k(\alpha)$. Using the partial fraction technique, we see that

$$\begin{aligned} \Pi_k(\alpha) \quad &= \quad \frac{1}{\alpha} \sum_{\ell=1}^{k} \left[\prod_{j=1, j\neq\ell}^{k} \frac{\lambda_{j-1}}{(\lambda_{j-1} - \lambda_{\ell-1}) + (j-\ell)\mu} \right] \frac{\lambda_{\ell-1}}{\lambda_{\ell-1} + \ell\mu + \alpha} \\ &\quad + \sum_{m=1}^{k} \frac{\mathbb{P}(Q(0) \ge m)}{\lambda_{m-1}} \sum_{\ell=m}^{k} \left[\prod_{j=m, j\neq\ell}^{k} \frac{\lambda_{j-1}}{(\lambda_{j-1} - \lambda_{\ell-1}) + (j-\ell)\mu} \right] \frac{\lambda_{\ell-1}}{\lambda_{\ell-1} + \ell\mu + \alpha} \end{aligned}$$

and inverting further yields

$$\begin{aligned}\mathbb{P}(Q(t) \geq k) &= \sum_{\ell=1}^{k} \left[\prod_{j=1, j\neq \ell}^{k} \frac{\lambda_{j-1}}{(\lambda_{j-1} - \lambda_{\ell-1}) + (j-\ell)\mu} \right] \frac{\lambda_{\ell-1}}{\lambda_{\ell-1} + \ell\mu} (1 - e^{-(\lambda_{\ell-1}+\ell\mu)t}) \\ &+ \sum_{m=1}^{k} \frac{\mathbb{P}(Q(0) \geq m)}{\lambda_{m-1}} \sum_{\ell=m}^{k} \lambda_{\ell-1} \left[\prod_{j=m, j\neq\ell}^{k} \frac{\lambda_{j-1}}{(\lambda_{j-1} - \lambda_{\ell-1}) + (j-\ell)\mu} \right] e^{-(\lambda_{\ell-1}+\ell\mu)t}.\end{aligned}$$

This proves the claim. □

3.3 The case where $\lambda_k = \lambda$, $\gamma_k = 0$

We finish the section by returning to a special case of the Bitcoin model from [11], where arrivals are assumed to occur in accordance to a Poisson process with rate λ. In our framework, this simply means that $\lambda_n := \lambda$ for each $n \geq 0$. This simplification yields a simpler expression for the probability mass function of $Q(t)$, as Corollary 3.1, an immediate consequence of Theorem 3.3, illustrates.

Corollary 3.1 *For each real number $t \geq 0$, we have for each integer $k \geq 1$ that*

$$\mathbb{P}(Q(t) \geq k) = \rho^{k-1} \sum_{\ell=1}^{k} \frac{(-1)^{\ell-1}}{(\ell-1)!(k-\ell)!} \frac{\lambda}{\lambda + \ell\mu} (1 - e^{-(\lambda+\ell\mu)t}) + \sum_{\ell=1}^{k} \mathbb{P}(Q(0) \geq \ell) \frac{[\rho(1-e^{-\mu t})]^{k-\ell}}{(k-\ell)!} e^{-(\lambda+\ell\mu)t}.$$

Lemma 5 on page 8 of [11] explains that through the Kolmogorov forward equations, one can show that for each real number $t \geq 0$, and each integer $k \geq 0$,

$$\mathbb{P}(Q(t) = k) = C_{k,0} + \sum_{\ell=1}^{k+1} C_{k,\ell} e^{-(\lambda+\ell\mu)t}$$

where the coefficients $C_{k,\ell}$ satisfy the recursive scheme

$$\begin{aligned} C_{k,0} &= \mathbb{P}(Q(\infty) = k) \\ (k+1-\ell)C_{k,\ell} &= \rho C_{k-1,\ell} - \sum_{j=i-1}^{k-1} C_{j,\ell}, \qquad 1 \leq \ell \leq k \\ C_{k,k+1} &= \mathbb{P}(Q(0) = k) - \sum_{j=0}^{k} C_{k,j}. \end{aligned}$$

Fortunately, Corollary 3.1 allows us to compute these coefficients exactly: after some algebra, we find

$$C_{k,0} = \sum_{\ell=1}^{k} \frac{(-1)^{\ell-1}\lambda}{(\ell-1)!(\lambda+\ell\mu)} \left[\frac{\rho^{k-1}}{(k-\ell)!} - \frac{\rho^k}{(k+1-\ell)!} \right]$$

$$C_{k,\ell} = \frac{(-1)^{\ell-1}\lambda}{(\ell-1)!(\lambda+\ell\mu)}\left[\frac{\rho^k}{(k+1-\ell)!} - \frac{\rho^{k-1}}{(k-\ell)!}\right] + \mathbb{P}(Q(0)\geq \ell)\left[\frac{(\rho(1-e^{-\mu t}))^{k-\ell}}{(k-\ell)!} - \frac{(\rho(1-e^{-\mu t}))^{k+1-\ell}}{(k+1-\ell)!}\right]$$

for $1 \leq \ell \leq k$, and

$$C_{k,k+1} = \frac{(-1)^k \rho^k \lambda}{k!(\lambda+(k+1)\mu)} - \mathbb{P}(Q(0) \geq k+1).$$

We should note, however, that $C_{k,0}$ is also the long-run fraction of time the Bitcoin model spends in state k, so one should use the corresponding element from the stationary distribution to actually calculate $C_{k,0}$ numerically.

We close by giving a series representation for the generating function of $Q(t)$.

Corollary 3.2 *The generating function of $Q(t)$ is as follows:*

$$\mathbb{E}[z^{Q(t)}] = 1 + (z-1)\left[\lambda e^{\rho z}\sum_{\ell=1}^{\infty}\frac{(-\rho)^{\ell-1}}{(\ell-1)!(\lambda+\ell\mu)}(1-e^{-(\lambda+\ell\mu)t})z^{\ell-1} + e^{\rho(1-e^{-\mu t})z}\sum_{\ell=1}^{\infty}\mathbb{P}(Q(0)\geq\ell)e^{-(\lambda+\ell\mu)t}z^{\ell-1}\right].$$

Proof The generating function of $Q(t)$ can be computed using Corollary 3.1, as well as the elementary identity

$$\mathbb{E}[z^{Q(t)}] = 1 + (z-1)\sum_{k=1}^{\infty} z^{k-1}\mathbb{P}(Q(t)\geq k), \quad 0 < z \leq 1.$$

□

4 The M$_t$/G/∞-like Bitcoin Model

We close by analyzing the M$_t$/G/∞-like Bitcoin queuing system analogous to the G/M/∞-like Bitcoin queueing system introduced in [11], that operates under the FIFO-batch departures discipline described in the Introduction. Now customers arrive to the system in accordance to a nonhomogeneous Poisson process $\{A(t); t \geq 0\}$ with rate function $\lambda : [0,\infty) \to [0,\infty)$. Letting $\{T_n\}_{n\geq 1}$ denote the points associated with $\{A(t); t\geq 0\}$, in that T_n denotes the time of the nth arrival to the queueing system, we associate with each T_n the random variable B_n, which denotes the amount of work brought by the nth arrival. We assume $\{B_n\}_{n\geq 1}$ is an i.i.d. sequence of nonnegative random variables with cumulative distribution function (cdf) F, and that $\{B_n\}_{n\geq 1}$ is independent of $\{T_n\}_{n\geq 1}$ (and, hence, $\{A(t); t \geq 0\}$). Readers should note also that any usage of B will refer to a generic random variable having cdf F.

Theorem 4.1 *Suppose that at time zero, the distributions of the amounts of work found in the system are i.i.d with cdf F, and independent of everything else. Then the distribution of $Q(t)$ is as follows: for each integer $k \geq 1$,*

$$\begin{aligned}\mathbb{P}(Q(t)\geq k) &= \sum_{m=1}^{k}\mathbb{P}(Q(0)\geq m)\mathbb{P}(B>t)^m\frac{\left(\int_0^t \mathbb{P}(B>t-u)\lambda(u)ds)\right)^{k-m} e^{-\int_0^t\lambda(u)du}}{(k-m)!} \\ &+ \int_0^t \mathbb{P}(B>t-s)e^{-\int_s^t\lambda(u)du}\frac{\left(\int_s^t\mathbb{P}(B>t-u)\lambda(u)du\right)^{k-1}}{(k-1)!}\lambda(s)ds.\end{aligned}$$

Obviously, modifications can be made to the distributions of work possessed by customers present at time zero: the only property we really need is that these amounts of work are independent of both $Q(0)$, as well as all future arrivals to the system.

Proof This result can be proven by making use of two-dimensional Poisson processes, combined with a slight modification of an argument that can be used to establish time-dependent versions of the distributional Little's law: for more on the time-dependent distributional Little's law, as well as time-dependent Little laws in general, see Bertsimas and Mourtzinou [4] as well as Fralix and Riaño [10]. Two-dimensional Poisson processes have been used previously to analyze the classical $\mathrm{M}_t/\mathrm{G}/\infty$ queue: see e.g. Eick et al [8] for more information on this topic.

First, observe that the coordinates $\{(T_n, B_n)\}_{n\geq 1}$ are simply points of a Poisson process on $[0,\infty)\times[0,\infty)$, whose mean measure is given by μ (defined on the Borel sets of $[0,\infty)\times[0,\infty)$) where for each $0 \leq s < t$, and each $0 \leq a < b$,

$$\mu((s,t]\times(a,b]) = \int_{(s,t]}\int_{(a,b]} \lambda(u)F(dy)du.$$

For each $0 \leq s < t$, we define the random variables $N^d((s,t]]$ and $N^q((s,t])$ as

$$N^d((s,t]] := \sum_{n=1}^{\infty} \mathbf{1}((T_n, B_n) \in \{(x,y) : s < x \leq t, y \leq t - x\})$$

$$N^q((s,t]] := \sum_{n=1}^{\infty} \mathbf{1}((T_n, B_n) \in \{(x,y) : s < x \leq t, y > t - x\})$$

and since $\{(T_n, B_n)\}_{n\geq 1}$ are points of a Poisson process, we can say that $N^d((s,t])$ and $N^q((s,t])$ are independent Poisson random variables having means that satisfy

$$\mathbb{E}[N^d((s,t])] = \int_{(s,t]} \mathbb{P}(B \leq t - x)\lambda(x)dx, \qquad \mathbb{E}[N^q((s,t])] = \int_{(s,t]} \mathbb{P}(B > t - x)\lambda(x)dx.$$

The next step of the proof involves determining how the FIFO-batch departures discipline affects customers present in the system at time zero. Throughout, we assume that each customer present in the system receives a unique number between 1 and $Q(0)$, and that whenever customer k from this group has its service completed, it, along with all other higher-labeled customers still in the system at that moment immediately leave the system. This means that the customer present at time zero with number k is placed in slot $Q(0) - k + 1$ at time zero. Once this rule has been specified, it is now clear how the FIFO-batch departures discipline operates over the entire time interval $[0,\infty)$.

For each integer $m \in \{1, 2, \ldots, Q(0)\}$, let $B_{m,0}$ denote the amount of work present by the customer at time 0 that has been given label m, and for each real number $s > 0$, let $B(s)$ denote the amount of work the customer arriving at time s brings to the system. Then for each integer $k \geq 1$, a bit of thought reveals that

$$\begin{aligned}\mathbf{1}(Q(t) \geq k) &= \sum_{m=1}^{k} \mathbf{1}(Q(0) \geq m)\left[\prod_{\ell=1}^{m} \mathbf{1}(B_{\ell,0} > t)\right]\mathbf{1}(N^d(0,t] = 0)\mathbf{1}(N^q((0,t]) = k - m) \\ &+ \int_0^t \mathbf{1}(B(s) > t - s)\mathbf{1}(N^d((s,t]) = 0)\mathbf{1}(N^q((s,t]) = k - 1)A(ds). \qquad (19)\end{aligned}$$

Identity (19) is a slight modification of an identity that can be used to derive the time-dependent version of the distributional Little's law. There are at least k customers in the system at time t if and only if either (a) there is a customer with label m from the group of customers present at time zero that is still present at time t, and this is only possible if customers $m-1, m-2, \ldots, 1$ are still present at time t, no customers arrive in $(0,t]$ and leave before time t, and exactly $k-m$ customers arrive in $(0,t]$ and are still present at t, or (b) there exists a customer that arrives at some time $s \in (0,t]$ and brings an amount of work larger than $t-s$, no customers that arrive in the system in $(s,t]$ leave at or before time t, and all $k-1$ arrivals that arrive in $(s,t]$ are still present in the system at time t.

The next step consists of taking the expected value of both sides of (19). First, it is clear that

$$\begin{aligned} & \mathbb{E}\left[\sum_{m=1}^{k} \mathbf{1}(Q(0) \geq m) \left[\prod_{\ell=1}^{m} \mathbf{1}(B_{\ell,0} > t)\right] \mathbf{1}(N^d(0,t] = 0)\mathbf{1}(N^q((0,t]) = k-m)\right] \\ = & \sum_{m=1}^{k} \mathbb{P}(Q(0) \geq m)\mathbb{P}(B > t)^m \mathbb{P}(N^d((0,t]) = 0)\mathbb{P}(N^q((0,t]) = k-m) \\ = & \sum_{m=1}^{k} \mathbb{P}(Q(0) \geq m)\mathbb{P}(B > t)^m e^{-\int_0^t \lambda(u)du} \frac{\left(\int_0^t \mathbb{P}(B > t-u)\lambda(u)du\right)^{k-m}}{(k-m)!} \end{aligned} \tag{20}$$

where the last equality follows from combining the two exponential terms found in each term within the finite sum. Next, observe that by the Campbell-Mecke formula, as well as arguments similar to those used to establish (20), we find that

$$\begin{aligned} & \mathbb{E}\left[\int_0^t \mathbf{1}(B(s) > t-s)\mathbf{1}(N^d((s,t]) = 0)\mathbf{1}(N^q((s,t]) = k-1)A(ds)\right] \\ = & \int_0^t \mathbb{P}(B > t-s) e^{-\int_s^t \lambda(u)du} \frac{\left(\int_s^t \mathbb{P}(B > t-u)\lambda(u)du\right)^{k-1}}{(k-1)!} \lambda(s)ds. \end{aligned} \tag{21}$$

Taking the expectation of both sides of (19), while applying both (20) and (21) to the right-hand-side yields

$$\begin{aligned} \mathbb{P}(Q(t) \geq k) = & \sum_{m=1}^{k} \mathbb{P}(Q(0) \geq m)\mathbb{P}(B > t)^m e^{-\int_0^t \lambda(u)du} \frac{\left(\int_0^t \mathbb{P}(B > t-u)\lambda(u)du\right)^{k-m}}{(k-m)!} \\ & + \int_0^t \mathbb{P}(B > t-s) e^{-\int_s^t \lambda(u)du} \frac{\left(\int_s^t \mathbb{P}(B > t-u)\lambda(u)du\right)^{k-1}}{(k-1)!} \lambda(s)ds \end{aligned}$$

which proves Theorem 4.1. □

It is not difficult to show that Theorem 4.1 agrees with Corollary 3.1 when arrivals are assumed to be homogeneous Poisson with rate λ, and services are assumed to be exponential with rate μ. Clearly, Theorem 4.1 can also be used to find the generating function of $Q(t)$, as well as expressions for the moments of $Q(t)$: we leave the details to the interested reader.

We should also mention that the stationary distribution of the M/G/∞-like Bitcoin infinite-server queue can be found using the techniques from this section, but the form of this distribution does not appear to be as elegant as the stationary distribution of the G/M/∞-like Bitcoin infinite-server queue, nor is it any more elegant than the expression given for the distribution of $Q(t)$ in Theorem 4.1, so we decided to omit it from the paper.

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