INTERSECTION DISJUNCTIONS FOR REVERSE CONVEX SETS

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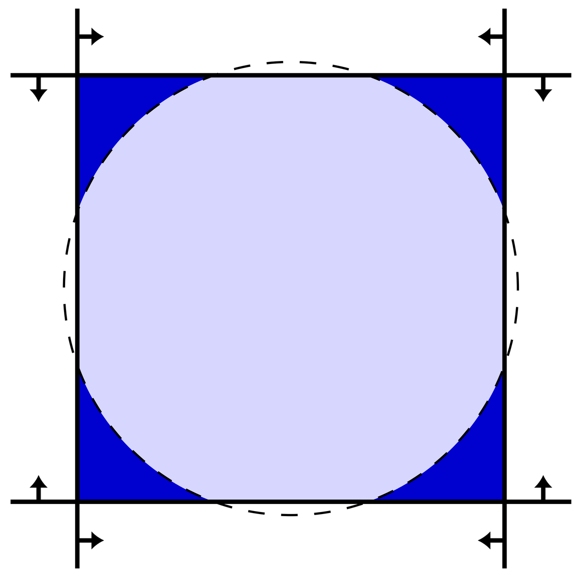
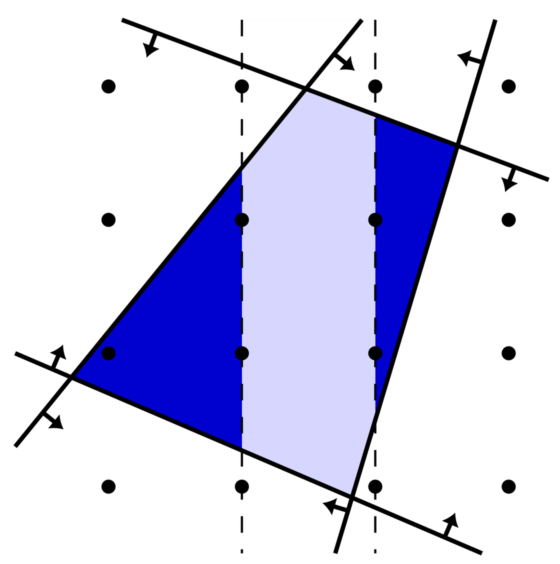
INTRODUCTION

- *Reverse convex sets*: of the form $P \setminus C$, where P is polyhedral and C is open and convex
- General set structure arising in MINLP
  - P: your favorite LP relaxation
  - C: contains no solutions feasible to MINLP
- We consider valid inequalities for $P \setminus C$
- Motivated by cases where C is nonpolyhedral or defined by a large number of inequalities





Reverse convex sets have applications to: polynomial optimization [4], bilevel optimization [5], binary integer programming [6], quadratically constrained programming [7], and DC programming [8]

INTERSECTION CUTS [1]

- Well-known method for generating valid inequalities for $P \setminus C$
- **Key:** requires simplicial cone with apex *in* C
  - e.g., cone formed at LP basic solution in C
- Let B be a basis for $P = \{x \in \mathbb{R}^n_+ : Ax = b\}$
  - N: nonbasic variables
  - $\bar{x}$: basic solution
  - $\bar{r}^j$: extreme rays defining basis cone at $\bar{x}$, $j \in N$
- P^B: LP basis cone formed by B (also a relaxation of P):

$$P^B = \Big\{\bar{x} + \sum_{j \in N} x_j \bar{r}^j : x_j \geq 0, j \in N\Big\}$$

- β_j: maximum we can move along $\bar{r}^j$ from $\bar{x}$ and stay in C

$$\beta_j := \sup\{\lambda \geq 0 : \bar{x} + \lambda \bar{r}^j \in C\} \qquad \forall j \in N$$

- **Theorem 1** (Balas). $\sum_{j \in N} \frac{x_j}{\beta_j} \geq 1$ *is valid for* $P \setminus C$.

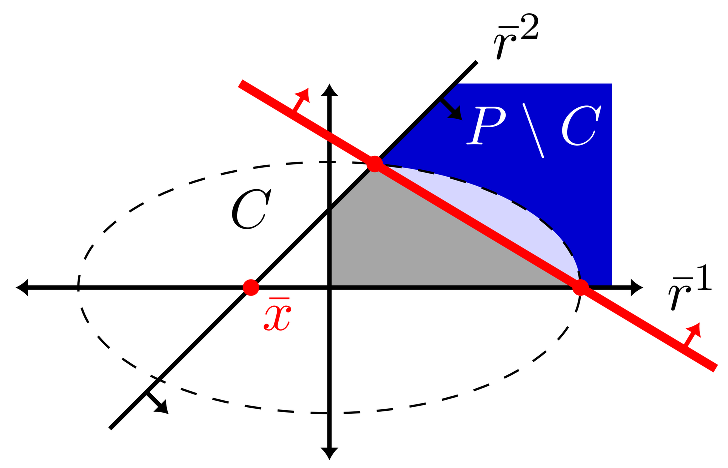


Figure 1: Intersection cut from infeasible basic solution *in* C

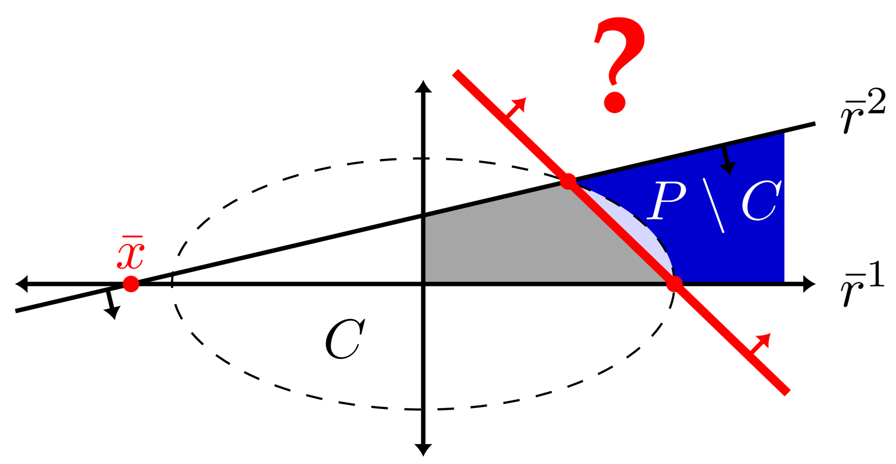


Figure 2: What if strong cut is generated by basic solution *outside* C?

TWO-WAY DISJUNCTION FOR $P \setminus C$

- Assume the basic solution is *not in* C
- α_j: minimum we can move along $\bar{r}^j$ from $\bar{x}$ and stay in C

$$\alpha_j := \inf\{\lambda \geq 0 : \bar{x} + \lambda \bar{r}^j \in C\} \qquad \forall j \in N$$

- Partition N into three sets depending on the extreme rays' intersection with Q:

$$N_0 := \{j \in N : \alpha_j = +\infty\}$$
$$N_1 := \{j \in N : \alpha_j < +\infty, \beta_j = +\infty\}$$
$$N_2 := \{j \in N : \alpha_j < +\infty, \beta_j < +\infty\},$$

Note: N_0 indices' rays do not intersect C

- **Theorem 2.** *If* $N_0 = \emptyset$, *every* $x \in P \setminus C$ *satisfies*

$$\sum_{j \in N} \frac{x_j}{\alpha_j} \leq 1 \text{ or } \sum_{j \in N} \frac{x_j}{\beta_j} \geq 1.$$

- Given Theorem 2's disjunction for $P \setminus C$, we generate cuts using a disjunctive program [2, 3]

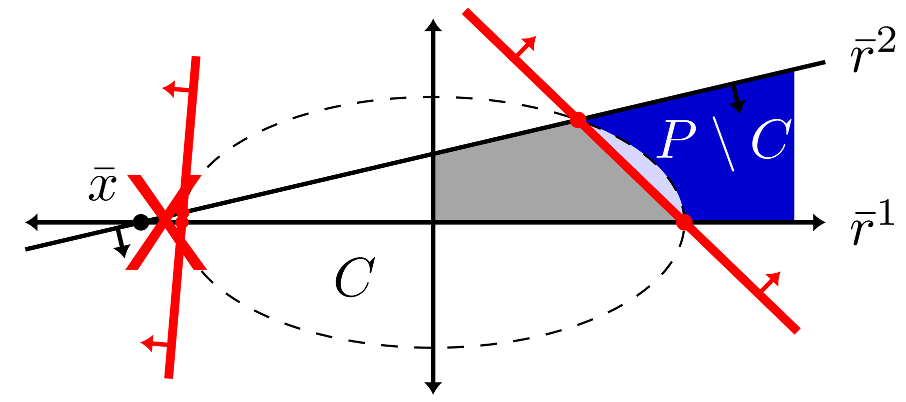


Figure 3: Theorem 2's two-way disjunction applied to Figure 1 finds strong cut

- **Corollary 3.** *If* $N_0 \neq \emptyset$ *but* $\bar{r}^j \in \operatorname{recc} C$ *for all* $j \in N_0$, *Theorem 2's disjunction contains* $P \setminus C$.
- **Idea:** how else can we use $\operatorname{recc} C$?

MULTI-TERM DISJUNCTION

- Line segment from $r \in \mathbb{R}^n$ given $\alpha, \beta \in \bar{\mathbb{R}}$:

$$(\alpha, \beta)r := \{\lambda r : \lambda \in (\alpha, \beta)\}$$

- Define $|N_2| + 1$ sets S_0^C and S_k^C $(k \in N_2)$:

$$S_0^C := \operatorname{conv}\Big(\bigcup_{j \in N_1 \cup N_2} (\alpha_j, +\infty)\bar{r}^j\Big) + \operatorname{recc} C$$

$$S_k^C := \operatorname{conv}\Big(\bigcup_{j \in N_2} [0, \beta_j)\bar{r}^j\Big) + (-\infty, 0]\bar{r}^k + \operatorname{recc} C$$

- **Lemma 4.** $P^B \setminus C \subseteq \bigcup_{k \in N_2 \cup \{0\}} (P^B \setminus S_k^C) =: D$
  - i.e., we can consider cuts for disjunction D
- **Theorem 5.** *It holds that*

$$\operatorname{clconv}(P^B \setminus C) = \operatorname{clconv}\Big(\bigcup_{k \in N_2 \cup \{0\}} (P^B \setminus S_k^C)\Big).$$

POLYHEDRAL RELAXATIONS

- To use Theorem 5 in a disjunctive program, we need *polyhedral relaxations* of

$$P^B \setminus S_k^C \quad \forall k \in N_2 \cup \{0\}$$

Valid inequalities for $P^B \setminus S_0^C$:

- M_0: indices $i \in N_1 \cup N_2$ such that

$$\big(\operatorname{cone}(\bar{r}^i, \bar{r}^j) \cap \operatorname{recc}(C)\big) \subsetneq [0, +\infty)\bar{r}^i \quad \forall j \in N_0$$

- For $j \in N_0$, $\gamma_j(U) := \max_{\gamma \geq 0} \gamma$

$$\text{s.t. } \gamma \bar{r}^j + \alpha_i \bar{r}^i \in \operatorname{recc}(C) + [0, +\infty)\bar{r}^i \quad \forall i \in U$$

- **Theorem 6.** *Let* $U \subseteq M_0$. *The inequality*

$$\sum_{j \in U} \frac{x_j}{\alpha_j} - \sum_{j \in N_0} \frac{x_j}{\gamma_j(U)} \leq 1$$

is valid for $P^B \setminus S_0^C$.

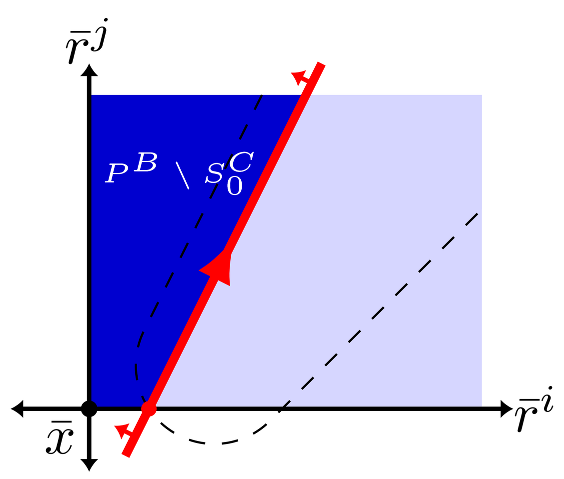


Figure 4: Theorem 6 gives valid inequalities for $P^B \setminus S_0^C$ by considering $\operatorname{recc}(C)$

We also provide inequalities for $P^B \setminus S_k^C$, $k \in N_2$

CONCLUSION

Our contributions:

- Propose disjunctive framework for generating cuts for $P \setminus C$ from points *outside* C
- Present two disjunctions for $P \setminus C$, one of which uses recession directions
- Provide linear relaxations of disjunctive terms

Future directions:

- Analyze disjunction strength for bounded P
- Finite/limit convergence to $\operatorname{conv}(P \setminus C)$?
- Generalize to bases without basic solution

ACKNOWLEDGMENTS

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REFERENCES

[1] E Balas. Intersection cuts: a new type of cutting planes for integer programming. *Operations Research*, 19(1):19–39, 1971.
[2] E Balas. Disjunctive programming. *Annals of Discrete Mathematics*, 5:3–51, 1979.
[3] E Balas. Disjunctive programming: Properties of the convex hull of feasible points. *Discrete Applied Mathematics*, 89(1–3):3–44, 1998. Originally MSRR no. 348, Carnegie Mellon University, July 1974.
[4] D Bienstock, C Chen, and G Muñoz. Outer-product-free sets for polynomial optimization and oracle-based cuts. *Preprint arXiv:1610.04604*, 2016.
[5] M Fischetti, I Ljubić, M Monaci, and M Sinnl. Intersection cuts for bilevel optimization. In *18th International Conference on Integer Programming and Combinatorial Optimization*, pages 77–88, 2016.
[6] M Raghavachari. On connections between zero-one integer programming and concave programming under linear constraints. *Operations Research*, 17(4):680–684, 1969.
[7] A Saxena, P Bonami, and J Lee. Convex relaxations of non-convex mixed integer quadratically constrained programs: extended formulations. *Mathematical Programming*, 124(1-2):383–411, 2010.
[8] H Tuy. A general deterministic approach to global optimization via d.c. programming. In *North-Holland Mathematics Studies*, volume 129, pages 273–303. 1986.