INTERSECTION DISJUNCTIONS FOR REVERSE CONVEX SETS

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INTRODUCTION

- Reverse convex sets: of the form \( P \setminus C \), where \( P \) is polyhedral and \( C \) is open and convex
- General set structure arising in MINLP
  - \( P \): your favorite LP relaxation
  - \( C \): contains no solutions feasible to MINLP
- We consider valid inequalities for \( P \setminus C \)
- Motivated by cases where \( C \) is nonpolyhedral or defined by a large number of inequalities

Reverse convex sets have applications to: polynomial optimization [4], bilevel optimization [5], binary integer programming [6], quadratically constrained programming [7], and DC programming [8]

TWO-WAY DISJUNCTION FOR \( P \setminus C \)

- Assume the basic solution is not in \( C \)
- \( \alpha_j \): minimum we can move along \( \tilde{p}_j \) from \( \tilde{x} \) and stay in \( C \):
  \[ \alpha_j := \inf \{ \lambda > 0 : \tilde{x} + \lambda \tilde{p}_j \in C \} \quad \forall j \in N \]
- Partition \( N \) into three sets depending on the extreme rays’ intersection with \( Q \):
  \[ N_0 := \{ j \in N : \alpha_j = +\infty \} \]
  \[ N_1 := \{ j \in N : \alpha_j < +\infty, \beta_j = +\infty \} \]
  \[ N_2 := \{ j \in N : \alpha_j < +\infty, \beta_j < +\infty \} \]
- Note: \( N_0 \)’s rays do not intersect \( C \)
- **Theorem 2.** If \( N_0 = \emptyset \), every \( x \in P \setminus C \) satisfies
  \[ \sum_{j \in N} \frac{x_j}{\alpha_j} \leq 1 \quad \text{or} \quad \sum_{j \in N} \frac{x_j}{\beta_j} \geq 1. \]
- **Corollary 3.** If \( N_0 \neq \emptyset \) but \( \tilde{p}_j \in \text{recc} C \) for all \( j \in N_0 \), Theorem 2’s disjunction contains \( P \setminus C \).
- Idea: how else can we use \( \text{recc} C \)?

MULTI-TERM DISJUNCTION

- Line segment from \( r \in \mathbb{R}^n \) given \( \alpha, \beta \in \mathbb{R} \):
  \[ (\alpha, \beta) := \{ (\lambda x, \lambda \beta) : \lambda \in [\alpha, \beta] \} \]
- Define \( |N_j| + 1 \) sets \( S_{0}^{\beta}(k) \) and \( S_{1}^{\beta}(k) \) (in \( N \)):
  \[ S_{0}^{\beta}(k) := \text{conv} \left( \bigcup_{j \in N \setminus N_2} (\alpha_j, +\infty) \tilde{p}_j \right) + \text{recc} C \]
  \[ S_{1}^{\beta}(k) := \text{conv} \left( \bigcup_{j \in N_2} [0, \beta_j] \tilde{p}_j \right) + (-\infty, 0] \tilde{p}_k + \text{recc} C \]
- **Lemma 4.** \( L \in C \subseteq \bigcup_{k \in N_2} (P \setminus C)(P \setminus S_{1}^{\beta}(k)) =: D \)
  - i.e., we can consider cuts for disjunction \( D \)
- **Theorem 5.** It holds that
  \[ \text{conv} (P \setminus C) = \text{conv} \left( \bigcup_{k \in N_2} (P \setminus S_{1}^{\beta}(k)) \right). \]

CONCLUSION

Our contributions:
- Propose disjunctive framework for generating cuts for \( P \setminus C \) from points outside \( C \)
- Present two disjunctions for \( P \setminus C \), one of which uses recursion directions
- Provide linear relaxations of disjunctive terms
- Future directions:
  - Analyze disjunction strength for bounded \( P \)
  - Finite/limit convergence to \( \text{conv}(P \setminus C) \)
  - Generalize to bases without basic solution

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REFERENCES


POLYHEDRAL RELAXATIONS

- To use Theorem 5 in a disjunctive program, we need polyhedral relaxations of \( P \setminus S_{0}^{\beta}(k) \)
- Valid inequalities for \( P \setminus S_{0}^{\beta}(k) \): \( M_{0} \) indices \( i \in N \setminus N_2 \) such that \( \{ \text{cone}(\tilde{p}_i, \tilde{p}_j) \cap \text{recc}(C) \} \subseteq [0, +\infty) \tilde{p}_j \) s.t. \( \beta_j = +\infty \)
- For \( j \in N_0 \), \( \gamma_j(U) := \max_{\tilde{p}_j \geq 0} \gamma_j \)
- **Lemma 4.** \( \tilde{p} \subseteq \text{conv}(\tilde{p}) \)
- **Theorem 6.** Let \( U \subseteq M_0 \). The inequality
  \[ \sum_{j \in U} \frac{x_j}{\alpha_j} - \sum_{j \notin N_2} \frac{x_j}{\gamma_j(U)} \leq 1 \]
  is valid for \( P \setminus S_{0}^{\beta}(k) \).