Cutting planes for Linear Programming with Complementarity Constraints

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Introduction

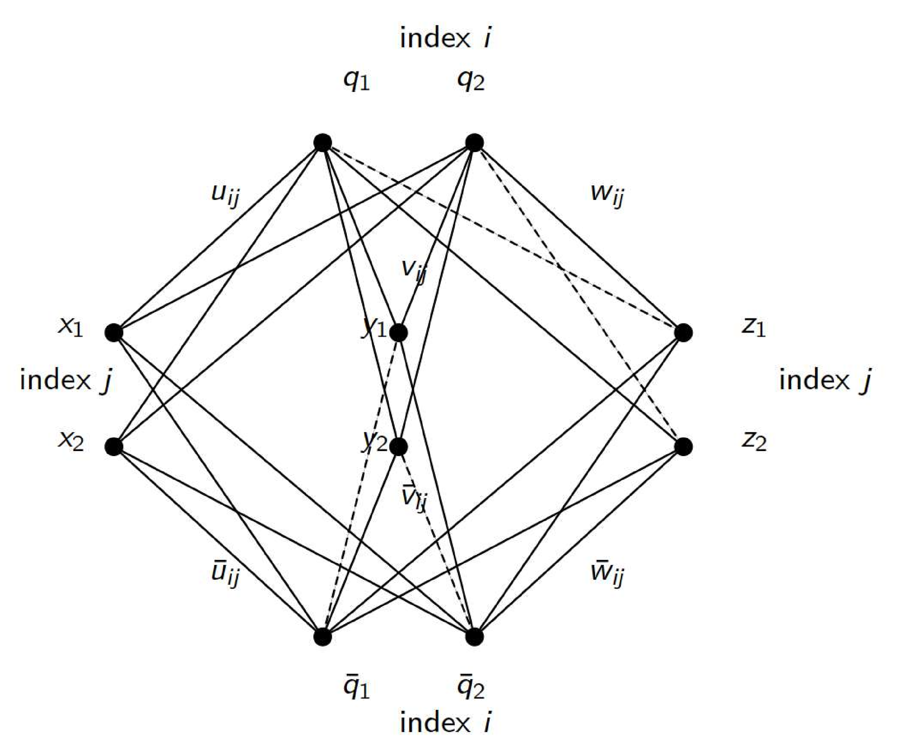
We consider linear programs with complementarity constraints in the form:

$$\text{(LPCC)}\quad \begin{array}{ll} \max & a^Tx + b^Ty + c^Tz \\ s.t. & Ax + By + Cz \leq d \\ & 0 \leq x, y, z \leq 1 \\ & y \perp z. \end{array} \tag{1}$$

- $\mathcal{C} = \{(y, z) \in \mathbb{R}^{2n} \mid y \perp z, 0 \leq y, z \leq 1\}$,
- $\mathcal{P} = \{(x, y, z) \in \mathbb{R}^{d+2n} \mid Ax + By + Cz \leq d, 0 \leq x, y, z \leq 1\}$,
- $\mathcal{F} = \{(x, y, z) \in \mathcal{P} | (y, z) \in \mathcal{C}\} = \{(x, y, z) \in \mathbb{R}^{d+2n} \mid Ax+By+Cz \leq d, 0 \leq x, y, z \leq 1, y \perp z\}$

▷ We're looking for tighter relaxation for $\text{conv}(\mathcal{F})$;
▷ $\text{conv}(\mathcal{C})$ can be precisely characterized, so need to consider $\mathcal{P}$ to improve relaxation for $\mathcal{F}$.

Graph G



This graph represent all the quadratic relationships between variables from **RLT**.

RLT [1]

- Add extra variables $q^i, \bar{q}^i, i \in [n]$ defined by:

$$q^i := \begin{cases} 1 & \text{if } y_i + z_i = 0 \\ \frac{y_i}{y_i+z_i} & \text{otherwise,} \end{cases},$$

$$\bar{q}^i := 1 - q^i = \begin{cases} 0 & \text{if } y_i + z_i = 0 \\ \frac{z_i}{y_i+z_i} & \text{otherwise.} \end{cases}$$

- Multiply each original constraints with $q^i, \bar{q}^i$, which gives us new lifted variables:

$$\begin{array}{llll} u^i = q^i x, & v^i = q^i y, & w^i = q^i z & \forall i \in [n] \\ \bar{u}^i = \bar{q}^i x, & \bar{v}^i = \bar{q}^i y, & \bar{w}^i = \bar{q}^i z & \forall i \in [n]. \end{array} \tag{Q}$$

- The complementarity relationship is captured by:

$$\begin{array}{rl} w^i_i = 0, \bar{v}^i_i = 0 & \forall i \in [n] \\ x = u^i + \bar{u}^i, y = v^i + \bar{v}^i, z = w^i + \bar{w}^i & \forall i \in [n] \\ q^i + \bar{q}^i = 1 & \forall i \in [n] \end{array} \tag{E}$$

"Curvy set"

Our new extended formulation comes from $\mathcal{S}$:

$$\mathcal{S} = \{(x, y, z, q, \bar{q}, u, \bar{u}, v, \bar{v}, w, \bar{w}) | \text{(Q), (E)}, (I)\} \tag{4}$$

Proposition 0.2: ideal polyhedral

The extreme points of $\mathcal{S}$ are contained in $\{0, 1\}^{d+4n+2dn+4n^2}$. Hence $conv(\mathcal{S})$ is polyhedral;

Extended formulation

McCormick relaxation on bilinear term (Q):

$$u^i_j \leq q^i, u^i_j \leq x_j, u^i_j \geq q^i + x_j - 1, u^i_j \geq 0, etc. \tag{M}$$

Multiply each constraint in $\mathcal{P}$ with $q^i, \bar{q}^i$ and replace each quadratic term with above variables:

$$Au^i + Bv^i + Cw^i \leq dq^i, A\bar{u}^i + B\bar{v}^i + C\bar{w}^i \leq d\bar{q}^i \quad \forall i \in [n] \tag{R}$$

$$0 \leq u^i, v^i, w^i \leq q^i, 0 \leq \bar{u}^i, \bar{v}^i, \bar{w}^i \leq \bar{q}^i \quad \forall i \in [n] \tag{I}$$

- McCormick relaxation is given by:

$$\mathcal{M} := \{(x, y, z, q, \bar{q}, u, \bar{u}, v, \bar{v}, w, \bar{w}) | \text{(M)}, (R), \text{(E)}\} \tag{2}$$

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$$\begin{aligned} conv(\mathcal{F}) &= conv(\cap^n_{i=1}((\mathcal{P} \cap \{y_i = 0\}) \cup (\mathcal{P} \cap \{z_i = 0\}))) \\ &\subseteq \cap^n_{i=1} conv((\mathcal{P} \cap \{y_i = 0\}) \cup (\mathcal{P} \cap \{z_i = 0\})) \\ &= \text{proj}_{x,y,z}\, \mathcal{B} \end{aligned} \tag{3}$$

Lemma 0.1

Denote $\mathcal{B}$ to be the Balas disjunctive formulation in lifted space, then:

$$\mathcal{B} = \mathcal{M};$$

Cutting planes from BQP

The Boolean Quadric Polytope corresponding to G:

$$\mathcal{BQP}(G) = conv\{(x, y, z, q, \bar{q}, u, \bar{u}, v, \bar{v}, w, \bar{w}) \in \{0, 1\}^{d+4n+2dn+4n^2} \mid \text{(Q)}\}.$$

From Proposition 0.2 we can actually have:

$$\begin{aligned} conv\mathcal{S} &= conv\{(x, y, z, q, \bar{q}, u, \bar{u}, v, \bar{v}, w, \bar{w}) \in \{0, 1\}^{d+4n+2dn+4n^2} \mid \text{(Q), (E)}, (I)\} \\ &= conv\{(x, y, z, q, \bar{q}, u, \bar{u}, v, \bar{v}, w, \bar{w}) \in \{0, 1\}^{d+4n+2dn+4n^2} \mid \text{(Q), (E)}\} \end{aligned}$$

Proposition 0.1

$conv(\mathcal{S}) = \mathcal{BQP}(G) \cap \text{(E)}$.

- Cutting planes from $\mathcal{BQP}(G)$ may be very useful for $\mathcal{F}$;
- But **ONLY** cutting planes from $\mathcal{BQP}(G)$ will be useful!

Preliminary Results

	Big-M	McCormick relaxation	McCormick relaxation + 6OCI
p0033-10	2579.96	2595.82	2616.92=OPT
sts45-4	15	22.173	22.177=OPT
sts45-6	15	22.3059	22.314=OPT

Future work

- Generalize our method to LPCC problems with no specific variable bounds;
- Incorporate special structure on B, C matrix (like in KKT condition);

Reference

[1] Trang T Nguyen, Mohit Tawarmalani, and Jean-Philippe P Richard. Convexification techniques for linear complementarity constraints. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 336–348. Springer, 2011.