Two-stage Stochastic *p*-order Conic Mixed Integer Programs: Tight Second Stage Formulations

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Two-stage Stochastic Conic MIP

Two-Stage Stochastic Conic Mixed Integer Program (TSS-CMIP)

$$\min\{cx + \mathbb{E}_{\xi_P}[\overline{\mathcal{Q}}_\omega(x)] : Ax \geqslant b, x \in \mathbb{Z}^{n_1}\}$$

where random variable ξ_P follows a known probability distribution P, and for scenario ω from a finite sample space Ω,

$$\overline{\mathcal{Q}}_\omega(x) := \min\Big\{g_\omega y_\omega + \sum_{j\in J} \hat{g}^j_\omega d^j_{\omega,0} : W_\omega y_\omega \geqslant r_\omega - T_\omega x,$$
$$\left\|E^j_\omega y_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0},\ j \in J$$
$$y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q-q_1}, d^j_{\omega,0} \in \mathbb{R}_+, j \in J\Big\}$$

Special cases of TSS-CMIP:

- Two-stage stochastic mixed integer program (TSS-MIP): TSS-CMIP with $J = \emptyset$
- TSS-MIP with quadratic objective function in the second stage

Extensive Formulation for TSS-CMIP (Large Scale CMIP)

$$\min\Big\{cx + \sum_{\omega\in\Omega} \overline{p}_\omega \Big(g_\omega y_\omega + \sum_{j\in J} \hat{g}^j_\omega d^j_{\omega,0}\Big) : Ax \geqslant b, T_\omega x + W_\omega y_\omega \geqslant r_\omega, \omega \in \Omega,$$
$$\left\|E^j_\omega y_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \qquad j \in J, \omega \in \Omega,$$
$$x \in \mathbb{Z}^{n_1}, y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q-q_1}, d^j_{\omega,0} \in \mathbb{R}_+, \qquad j \in J, \omega \in \Omega\Big\}$$

Note:

- We use $\|\cdot\|_p$ to denote l_p-norm, i.e. $\|\mathbf{y}\|_p = (\sum_k |y_k|^p)^{1/p}$ for $p \geqslant 1$.
- Define $\overline{p}_\omega$ as the probability of occurrence of scenario $\omega \in \Omega$.

Research Contributions

- Provide sufficient conditions to convexify second stage CMIPs of TSS-CMIPs.
- Present TSS-CMIPs with structured p-order CMIPs in the second stage and derive classes of parametric (non)-linear cuts which satisfying above conditions.
- Perform computational results for TSS-CMIPs with two structured CMIPs in the second stage, i.e., $E^j_\omega = \mathbf{I}$ or E^j_ω is a randomly generated Totally Unimodular (TU) matrix:
  - The number of integer variables in the extensive formulation are significantly reduced (for example, 300025 to 25) by adding parametric (non)-linear cuts a priori.
  - CPLEX 12.70 with its default settings (without our parametric cuts) could not solve 110 out of 210 randomly generated instances within 3 hours time limit and 24 GB RAM.
  - Whereas after adding our parametric cuts, 107 out of the above 110 instances can be solved in 8.2 minutes (on average).
  - Parametric cuts closed the integrality gap by 31.48% (on average).

Second Stage Convexification Approach

- Denote the feasible region of the extensive formulation and the second stage program for $\omega \in \Omega$ by $\overline{\mathcal{P}}$ and $\overline{\mathcal{K}}_\omega(x)$, respectively.
- Define the partial convex hull of a set $\mathcal{Z}$ as $\mathcal{Z}_{pch}$, where $\mathcal{Z}_{pch}$ has lesser number of integrality constraints (but possibly more linear or nonlinear inequalities) than $\mathcal{Z}$, and $\mathcal{Z} \subseteq \mathcal{Z}_{pch} \subseteq conv(\mathcal{Z}) = conv(\mathcal{Z}_{pch})$.
- Let

$$\overline{\mathcal{K}}^\omega_{tight}(x) := \{(y_\omega, d_\omega) \in \mathbb{R}^q \times \mathbb{R}^{|J|}_+ : W_\omega y_\omega \geqslant r_\omega - T_\omega x,$$
$$\left\|E^j_\omega y_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \qquad j \in J,$$
$$\left\|\overline{E}^j_\omega y_\omega + \overline{F}^j_\omega x - \overline{h}^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \qquad j \in J\},$$

where $\overline{E}^j_\omega, \overline{F}^j_\omega$, and $\overline{h}^j_\omega$ are known matrices and vectors for $(x,\omega) \in (X, \Omega)$.

Theorem 1. If $conv(\overline{\mathcal{K}}_\omega(x)) = \mathrm{Proj}_y(\overline{\mathcal{K}}^\omega_{tight}(x)))$ for all $x \in X = \{x : Ax \geqslant b, x \in \mathbb{Z}^{n_1}\}$ and $\omega \in \Omega$, then $\mathrm{Proj}_{x,y}(\overline{\mathcal{P}}_{tight})$ is a partial convex hull of $\overline{\mathcal{P}}$, where

$$\overline{\mathcal{P}}_{tight} := \Big\{(x, y_\omega, d_\omega) \in X \times \mathbb{R}^q \times \mathbb{R}^{|J|}_+ : T_\omega x + W_\omega y_\omega \geqslant r_\omega, \qquad \omega \in \Omega$$
$$\left\|E^j_\omega y_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \qquad j \in J, \omega \in \Omega,$$
$$\left\|\overline{E}^j_\omega y_\omega + \overline{F}^j_\omega x - \overline{h}^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \qquad j \in J, \omega \in \Omega\Big\}$$

Two-Stage Distributionally Robust CMIPs

Our results for TSS-CMIPs are also applicable for the TSDR-CMIP, defined as

$$\min\Big\{cx + \max_{P\in\mathfrak{P}} \mathbb{E}_{\xi_P}[\mathcal{Q}_\omega(x)] \Big| Ax \geqslant b, x \in \mathbb{Z}^{n_1}\Big\}$$

where complete information about the probability distribution P followed by the random variable ξ_P is not known but it belongs to a set of distributions $\mathfrak{P}$.

Structured CMIPs (Atamtürk and Narayanan, 2008)

Conic Mixed Integer Rounding (CMIR) Cuts

- A single-constraint conic mixed integer set with one integer variable is defined as $\overline{Z} := \left\{(\sigma, v, \rho_0) \in \mathbb{Z} \times \mathbb{R}^2_+ : \sqrt{(\sigma-\beta)^2 + v^2} \leqslant \rho_0\right\}$, where $\beta \in \mathbb{R}$.
- Atamtürk and Narayanan reformulated set $\overline{Z}$ using additional continuous variables to get

$$Z := \left\{(\sigma, v, \rho_0, \rho_1, \rho_2) \in \mathbb{Z} \times \mathbb{R}^4_+ : |\sigma - \beta| \leqslant \rho_1, |v| \leqslant \rho_2, \sqrt{\rho_1^2 + \rho_2^2} \leqslant \rho_0\right\},$$

- They obtained the convex hull of $Z_1 := \{(\sigma, \rho_1) \in \mathbb{Z} \times \mathbb{R}_+ : |\sigma - \beta| \leqslant \rho_1\}$ can be obtained by adding

$$\left(1 - 2\beta^{(1)}\right)(\sigma - \lfloor\beta\rfloor) + \beta^{(1)} \leqslant \rho_1$$

where $\beta^{(1)} = \beta - \lfloor\beta\rfloor$, to the continuous relaxation of Z_1.
- We generalize set Z by studying TSS-CMIPs with multi-constraints p-order conic mixed integer sets having multiple integer variables in the second stage.

Structured TSS-CMIPs

Theorem 2. Let

$$\overline{\mathcal{Q}}_\omega(x) := \min\Big\{g_\omega y_\omega + \sum_{j\in J} \hat{g}^j_\omega d^j_{\omega,0} : W_\omega y_\omega \geqslant r_\omega - T_\omega x,$$
$$\left\|\mathbf{1}y^j_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, y^j_\omega \in \mathbb{Z}, d^j_{\omega,0} \in \mathbb{R}_+, j \in J\Big\},$$

where W_ω is a TU matrix and r_ω is integral. The convex hull of the feasible region of $\overline{\mathcal{Q}}_\omega(x)$ for all $x \in X$ is given by

$$\Big\{(y_\omega, d_{\omega,0}) \in \mathbb{R}^{|J|} \times \mathbb{R}^{|J|}_+ : \ W_\omega y_\omega \geqslant r_\omega - T_\omega x,$$
$$\left\|\mathbf{1}y^j_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \left\|(\mathbf{1} - 2\mu^j_\omega)y^j_\omega + \overline{F}^j_\omega x - \overline{h}^j_\omega\right\|_p \leqslant d^j_{\omega,0}, j \in J\Big\},$$

where the ith row of $\mu^j_\omega, \overline{F}^j_\omega$, and $\overline{h}^j_\omega$ are denoted by $\mu^j_{\omega,i} = h^j_{\omega,i} - \left\lfloor h^j_{\omega,i}\right\rfloor$, $\overline{f}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right) f^j_{\omega,i}$, and $\overline{h}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right)\left\lfloor h^j_{\omega,i}\right\rfloor - \mu^j_{\omega,i}$, respectively.

Theorem 3. Let

$$\overline{\mathcal{Q}}_\omega(x) := \min\Big\{g_\omega y_\omega + \sum_{j\in J} \hat{g}^j_\omega d^j_{\omega,0} : W_\omega y_\omega \geqslant r_\omega - T_\omega x,$$
$$\left\|E^j_\omega y^j_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, y^j_\omega \in \mathbb{Z}^q, d^j_{\omega,0} \in \mathbb{R}_+, j \in J\Big\},$$

where $E^j_\omega, j \in J$, and W_ω are TU matrices and r_ω is integral. The convex hull of the feasible region of $\overline{\mathcal{Q}}_\omega(x)$ for all $x \in X$ is given by

$$\Big\{(y_\omega, d_{\omega,0}) \in \mathbb{R}^{q|J|} \times \mathbb{R}^{|J|}_+ : W_\omega y_\omega \geqslant r_\omega - T_\omega x,$$
$$\left\|E^j_\omega y^j_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \left\|\overline{E}^j_\omega y^j_\omega + \overline{F}^j_\omega x - \overline{h}^j_\omega\right\|_p \leqslant d^j_{\omega,0}, j \in J\Big\},$$

where the ith row of $\overline{E}^j_\omega, \overline{F}^j_\omega$, and $\overline{h}^j_\omega$ are denoted by $\overline{e}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right) e^j_{\omega,i}$, $\overline{f}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right) f^j_{\omega,i}$, and $\overline{h}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right)\left\lfloor h^j_{\omega,i}\right\rfloor - \mu^j_{\omega,i}$, respectively, and $\mu^j_{\omega,i} = h^j_{\omega,i} - \left\lfloor h^j_{\omega,i}\right\rfloor$. Furthermore, in higher dimensional space $\overline{\mathcal{Q}}_\omega(x)$ can be reformulated as:

$$\mathcal{Q}_\omega(x) := \min\Big\{g_\omega y_\omega + \sum_{j\in J} \hat{g}^j_\omega d^j_{\omega,0} : W_\omega y_\omega \geqslant r_\omega - T_\omega x$$
$$|e^j_{\omega,i} y^j_\omega + f^j_{\omega,i} x - h^j_{\omega,i}| \leqslant d^j_{\omega,i}, \left\|d^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \ i = 1, \ldots, m_2, \ j \in J,$$
$$y^j_\omega \in \mathbb{Z}^q, d^j_{\omega,0} \in \mathbb{R}_+, d^j_\omega \in \mathbb{R}^{m_2}_+ \qquad j \in J\Big\}.$$

Then, for all $x \in X$, the convex hull of the feasible region of $\mathcal{Q}_\omega(x)$, denoted by $\mathcal{K}_\omega(x)$, is obtained by adding $m_2 \times |J|$ number of the following linear inequalities (in the higher dimensional space) to the continuous relaxation of $\mathcal{K}_\omega(x)$:

$$\left(1 - 2\mu^j_{\omega,i}\right)\left(e^j_{\omega,i} y^j_\omega - \left\lfloor h^j_{\omega,i}\right\rfloor + f^j_{\omega,i} x\right) + \mu^j_{\omega,i} \leqslant d^j_{\omega,i}, \ i = 1, \ldots, m_2, \ j \in J.$$

Generalized Structured TSS-CMIP

We obtain the partial convex hull ($\overline{\mathcal{P}}_{tight}$) of the feasible region of the extensive formulation, i.e., $\overline{\mathcal{P}}$, when $E^j_\omega, j \in J$, W_ω are TU matrices and r_ω is integral, where

$$\overline{\mathcal{P}}_{tight} := \Big\{(x, \{y_\omega, d_{\omega,0}\}_{\omega\in\Omega}) \in X \times \mathbb{R}^{q|\Omega|} \times \mathbb{R}^{|\Omega|\times|J|}_+ : \ T_\omega x + W_\omega y_\omega \geqslant r_\omega,$$
$$\left\|E^j_\omega y^j_\omega + F^j_\omega x - h^j_\omega\right\|_p \leqslant d^j_{\omega,0}, \left\|\overline{E}^j_\omega y^j_\omega + \overline{F}^j_\omega x - \overline{h}^j_\omega\right\|_p \leqslant d^j_{\omega,0}, j \in J, \omega \in \Omega\Big\}$$

such that the ith row of $\mu^j_\omega, \overline{E}^j_\omega, \overline{F}^j_\omega$, and $\overline{h}^j_\omega$ are given by $\mu^j_{\omega,i} = h^j_{\omega,i} - \left\lfloor h^j_{\omega,i}\right\rfloor$, $\overline{e}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right) e^j_{\omega,i}, \overline{f}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right) f^j_{\omega,i}$, and $\overline{h}^j_{\omega,i} = \left(1 - 2\mu^j_{\omega,i}\right)\left\lfloor h^j_{\omega,i}\right\rfloor - \mu^j_{\omega,i}$, respectively.

Computational Results

- SCMIP.α.β.λ denotes instance category, where α is the number of integer variables in the first stage, β is the number of integer variables in the second stage, and λ is the number of second-order conic constraints in the second stage.
- We consider the extensive formulation of TSS-CMIP in higher dimensional space.
- NO-PCUTS: CPLEX 12.70 with its default settings solve the reformulated extensive formulation without our parametric cuts.
- WITH-PCUTS: CPLEX 12.70 with its default settings solve the reformulated extensive formulation with our parametric cuts added a priori.
- #IVar, #LCon, #PCuts: # of integer variables, linear constraints, parametric cuts.
- $G\% = 100 \times \left(V_{pcut} - V_{cp}\right) / \left(V_{mip} - V_{cp}\right)$, where V_{pcut}, V_{cp} and V_{mip} denote the optimal objective value of continous relaxation of extensive formulation with and without our parametric cuts, and extensive CMIP formulation, respectively.
- T-EF, T-EFC: time (in seconds).
- TL: Instances not solved within 3 hours; OM: Out of memory (24 GB RAM).
- Each entry in Table 1 and Table 2 corresponds to the average for five instances and three instances, respectively.
- In Table 1 and Table 2, the first 10 instances categories are motivated from SIPLIB TSS-MIP instances. The remaining instances have large number of scenarios.

Results of Computational Experiments for TSS-CMIPs with $E^j_\omega = \mathbf{I}$

Instance Category	$\lvert\Omega\rvert$	NO-PCUTS			WITH-PCUTS			
		#IVar	#LCon	T-EF	#IVar	#PCuts	T-EFC	G%
SCMIP.5.125.1	50	12505	14001	32.1	5	6250	5.4	35.4
SCMIP.5.125.1	100	25005	28001	282.9	5	40501	91.5	15.9
SCMIP.10.500.1	50	50010	53001	TL	10	25000	148	18.4
SCMIP.10.500.1	100	100010	106001	TL	10	50000	602	38.7
SCMIP.15.625.1	5	6265	7051	27.4	15	3375	1.40	3.4
SCMIP.15.625.1	15	18765	21151	116	15	10125	5.77	13.5
SCMIP.15.625.1	50	62515	70501	TL	15	33750	550	22.4
SCMIP.240.120.1	20	5040	4905	9.3	240	2400	1.00	4.7
SCMIP.240.120.1	50	12240	12255	51.9	240	6000	2.61	6.6
SCMIP.5.10.1	100	2005	2505	6.3	5	1000	1.26	27.7
SCMIP.5.10.1	500	10005	12505	29.3	5	5000	4.8	43.5
SCMIP.5.10.1	1000	20005	25005	67.4	5	10000	9.9	52.5
SCMIP.5.10.1	5000	100005	125005	284	5	50000	82.7	73
SCMIP.5.10.1	10000	200005	250005	TL	5	100000	219	85
SCMIP.10.10.1	100	2010	4005	0.79	10	1000	1.6	0.75
SCMIP.10.10.1	500	10010	20005	4.3	10	5000	5.4	2.1
SCMIP.10.10.1	1000	20010	40005	13	10	10000	14.6	10.6
SCMIP.10.10.1	5000	100010	200005	502	10	50000	91.8	13.1
SCMIP.10.10.1	10000	200010	400005	1490	10	100000	288	27.6

Results of Computational Experiments for TSS-CMIPs with $E^j_\omega = TU$

Instance Category	$\lvert\Omega\rvert$	NO-PCUTS			WITH-PCUTS			
		#IVar	#LCon	T-EF	#IVar	#PCuts	T-EFC	G%
SCMIP.5.125.3	50	37505	69001	TL	5	15000	22.9	22.9
SCMIP.5.125.3	100	75005	138001	TL	5	30000	73.5	73.5
SCMIP.10.500.3	50	150010	243001	TL	10	45000	330	13.4
SCMIP.10.500.3	100	300010	486001	TL	10	90000	1742	30.5
SCMIP.15.625.3	5	18765	31051	1069	15	6000	56.1	0.1
SCMIP.15.625.3	10	37515	62101	TL	15	12000	162	0
SCMIP.15.625.3	15	56265	93151	TL	15	18000	265	0.95
SCMIP.240.120.3	20	14640	26505	TL	240	6000	46	2.3
SCMIP.240.120.3	50	36240	66255	TL	240	15000	386	1.06
SCMIP.10.25.3	50	7510	11005	9.98	10	1500	1.8	71
SCMIP.10.25.3	100	15010	22005	20.9	10	3000	4.4	64
SCMIP.10.25.3	200	30010	44005	40.9	10	6000	10.3	97
SCMIP.10.75.3	50	22510	34005	OM	10	5250	19.6	29.1
SCMIP.10.75.3	100	45010	68005	OM	10	10500	41.1	48.1
SCMIP.10.75.3	200	90010	136005	TL	10	21000	121	77.2
SCMIP.25.50.3	50	15025	23510	OM	25	3750	5.89	30.2
SCMIP.25.50.3	200	60025	94010	OM	25	15000	54.8	57.5
SCMIP.25.50.3	500	150025	235010	OM	25	37500	569	84.3
SCMIP.25.100.3	50	30025	55010	165	25	12000	49	3.74
SCMIP.25.100.3	200	120025	220010	OM	25	48000	2428	5.35
SCMIP.25.100.3	500	300025	550010	OM	25	120000	4256	54.7

References

1. Atamtürk , A., Narayanan, V., Conic mixed-integer rounding cuts *Mathematical Programming* 122 (2008), pp.1-20.
2. Bansal, M., Zhang, Y., Two-stage stochastic (and distributaionally robust *p*-order conic mixed integer programs: Tight second stage formulations, *SIAM Journal on Optimization, under review* (2018)