Generalized Chvátal-Gomory closures for integer programs with bounds on variables

Sanjeeb Dash[1] Oktay Günlük[1] **Dabeen Lee**[2]

[1]IBM Thomas J. Watson Research Center [2]Tepper School of Business, Carnegie Mellon University

Carnegie Mellon University

Introduction

Motivation

Integer programming problems often have nonnegative or bounded variables. Chvátal-Gomory (CG) cuts form an important class of cutting planes for integer programs, but usually do not incorporate all available variable bounds. We consider a natural generalization of CG cuts that uses available bound information. We introduce the notion of closure defined by the generalized CG cuts.

Generalized Chvátal-Gomory cuts

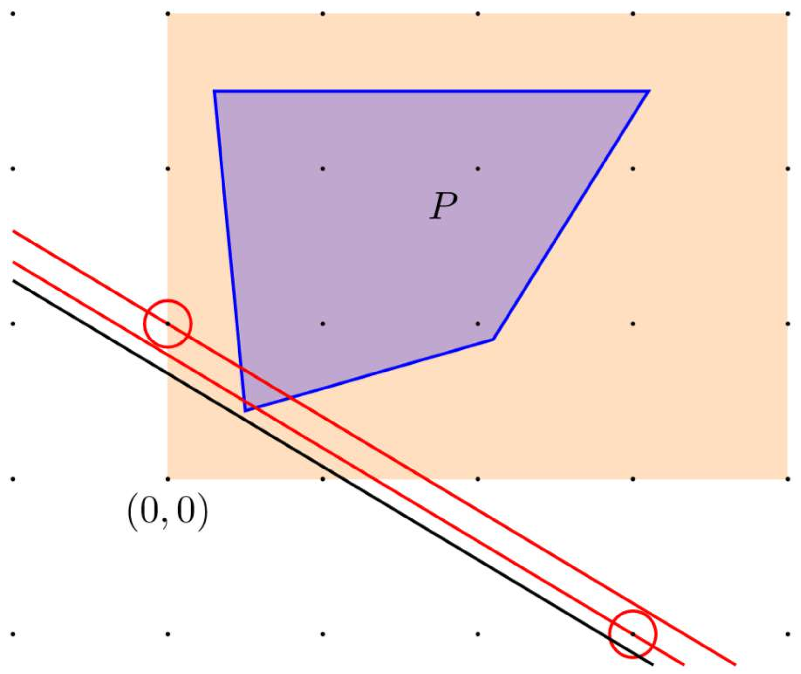
Let S be a subset of $\mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron contained in $\mathrm{conv}(S)$.

Given a valid inequality $cx \geqslant d$ for P with $c \in \mathbb{Z}^n$,

$$\mathbf{cx} \geqslant \min\{\mathbf{cz} : \ \mathbf{cz} \geqslant \mathbf{d}, \ \mathbf{z} \in \mathbf{S}\}$$

is called the *S-Chvátal-Gomory (S-CG) cut* obtained from $cx \geqslant d$.

For example, consider $S = \mathbb{Z}^2_+$ and $P \subseteq \mathbb{R}^2_+$.



Given a valid inequality $3x_1 + 5x_2 \geqslant 3.4$,

- CG cut: $3x_1 + 5x_2 \geqslant 4$,
- S-CG cut: $3x_1 + 5x_2 \geqslant 5$.

Generalized Chvátal-Gomory closure

The *S-Chvátal-Gomory (S-CG) closure* of P is defined as

$$\bigcap_{\mathbf{c} \in \mathbb{Z}^n} \left\{ \mathbf{x} : \ \mathbf{cx} \geqslant \min\left\{ \mathbf{cz} : \ \mathbf{cz} \geqslant \min\{\mathbf{cx} : \mathbf{x} \in \mathbf{P}\}, \ \mathbf{z} \in \mathbf{S} \right\} \right\},$$

that is, the set of points that satisfy all possible S-CG cuts for P.

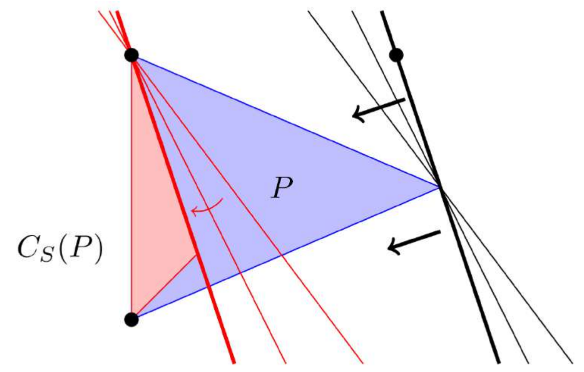
Question

Given a rational polyhedron P, we know that the CG closure of P is, again, a rational polyhedron. When is it that the S-CG closure of P is also a rational polyhedron?

Polyhedrality of the generalized Chvátal-Gomory closure

Differences between the CG closure and the S-CG closure

(1) Facets of the S-CG closure are not necessarily defined by S-CG cuts.



- An example: let $S = \{0,1\}^4$ and P be the convex hull of six points in $[0,1]^4$:

$$P = \mathrm{conv}\left\{ \left(\tfrac{1}{2}, 0, 0, 0\right), (1,0,0,0), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,1) \right\}.$$

- $2x_1 + x_2 + x_3 + x_4 \geqslant 2$ defines a facet of the S-CG closure of P.
- $2x_1 + x_2 + x_3 + x_4 \geqslant 1$ defines a supporting hyperplane of P that contains a point in S, so $2x_1 + x_2 + x_3 + x_4 \geqslant 2$ cannot be an S-CG cut.

(2) Let $cx \geqslant \beta$ be a valid inequality for P where c consists of relatively prime integers, and let $cx \geqslant \gamma$ be the S-CG cut derived from it. Although $\gamma - \beta \leqslant 1$ when $S = \mathbb{Z}^n$, $\gamma - \beta$ cannot be bounded by a fixed constant for general S (for example, $S = \mathbb{Z}^n_+$).
$*$ *The fact that $\gamma - \beta$ is bounded for CG cuts was crucial to prove the polyhedrality of the CG closure.*

Prior result (Dunkel and Schulz (2011))

Let $S = \{0,1\}^n$. If P is a rational polytope in $[0,1]^n$, then the S-CG closure of P is a rational polytope.

Theorem

Let P be a rational polyhedron and

$$S = \left\{ (z^1, z^2, z^3, z^4) \in T \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3} \times \mathbb{Z}^{n_4} : \begin{array}{c} \ell^2 \leqslant z^2 \\ z^3 \leqslant u^3 \end{array} \right\},$$

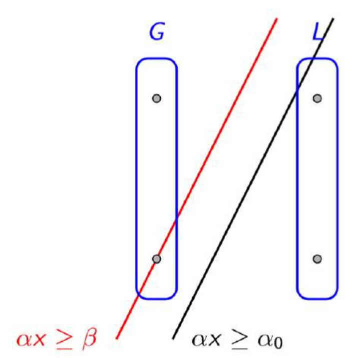
where T is a finite subset of $\mathbb{Z}^{n_1}$. If P is contained in $\mathrm{conv}(S)$, then the S-CG closure of P is a rational polyhedron.

Proof outline

Case 1: $\mathrm{conv}(S)$ is bounded

Dunkel and Schulz [1] considered the case when $S = \{0,1\}^n$, but their argument can be extended to the general bounded case.

- Let $\alpha x \geqslant \alpha_0$ be a valid inequality. If $\alpha x \geqslant \beta$ is the S-CG cut obtained from it, (α, β) satisfies the following for some $\lambda \geqslant 0$:



$$\begin{aligned} (\alpha, \alpha_0) &= (\lambda A, \lambda b), & \\ \alpha z &\geqslant \beta, & \forall z \in G \\ \alpha z &\leqslant \alpha_0 - \tfrac{1}{\Delta}, & \forall z \in L \\ \beta &\geqslant \alpha_0 \end{aligned}$$

where $P = \{x : \ Ax \leqslant b\}$.

- In fact, adding $\alpha x \geqslant \beta$ for every (α, β) that satisfies the system is sufficient to describe the S-CG closure.

Case 2: $\mathrm{conv}(S)$ is a cylinder

Next, we consider the case $S = T \times \mathbb{Z}^{n_2}$ where T is a finite subset of $\mathbb{Z}^{n_1}$.

Given a valid inequality $\alpha_1 x_1 + \alpha_2 x_2 \geqslant \alpha_0$, when is it that $\alpha_1 x_1 + \alpha_2 x_2 \geqslant \beta$ is the corresponding S-CG cut?

- If $\alpha_2 = 0$, then the condition can be stated similarly as in Case 1.
- If $\alpha_2 \neq 0$, what would β be? If g is the gcd of the components in α_2,

$$\beta = \min_{z_1 \in T} \left\{ \alpha_1 z_1 + g \left\lceil \frac{\alpha_0 - \alpha_1 z_1}{g} \right\rceil \right\}$$

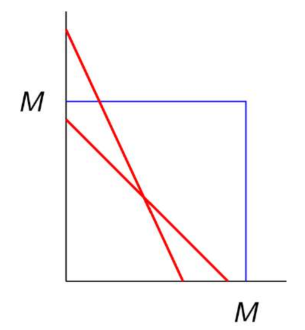
- If $\alpha_2 \neq 0$ and $\alpha_1 x_1 + \alpha_2 x_2 \geqslant \beta$ is not dominated by other S-CG cuts, then there exists $0 \leqslant \lambda \leqslant g\mathbf{1}$ such that $(\alpha_1, \alpha_2, \beta) = (\lambda A_1, \lambda A_2, \lambda b)$ where $P = \{(x_1, x_2) : A_1 x_1 + A_2 x_2 \leqslant b\}$.
Note that this argument is in spirit of Schrijver's technique [2].
- We can write a set of linear constraints in terms of $\alpha_1, \alpha_2, \alpha_0, \beta, g, \lambda$ that are sufficient to describe the condition.

Case 3: $\mathrm{conv}(S)$ is a half-cylinder

We consider the case $S = T \times \mathbb{Z}^{n_2}_+$ where T is a finite subset of $\mathbb{Z}^{n_1}_+$.

Given a valid inequality $\alpha_1 x_1 + \alpha_2 x_2 \geqslant \alpha_0$, when is it that $\alpha_1 x_1 + \alpha_2 x_2 \geqslant \beta$ is not dominated by other S-CG cuts?

- Either $\alpha_2 \geqslant 0$ or $\alpha_2 \leqslant 0$.
- There exists a sufficiently large constant M such that $0 \leqslant \frac{\alpha_{2j}}{\beta} \leqslant M$ for each j.



In this case, all the intercepts of $\{x : \alpha x = \beta\}$ are contained in a box.

$\rightarrow$ We can reduce this case to Case 1.

- We can write a set of linear constraints in terms of $\alpha_1, \alpha_2, \alpha_0, \beta$ to guarantee that $\alpha_1 x_1 + \alpha_2 x_2 \geqslant \beta$ is a non-redundant S-CG cut.

Case 4: general case

(1) After taking an appropriate unimodular transformation, we may assume that $S = T \times \mathbb{Z}^{n_2}_+ \times \mathbb{Z}^{n_3}$ where T is a finite subset of $\mathbb{Z}^{n_1}_+$.

(2) Consider the S_0-CG closure, where $S_0 = T \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$. It is a rational polyhedron by Case 2 and contains the S-CG closure as S_0 contains S.

(3) Given a valid inequality $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \geqslant \alpha_0$, when is it that the corresponding S-CG cut $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \geqslant \beta$ cuts off some part of the S_0-CG closure?

- $\alpha_3 = 0$.
- Either $\alpha_2 \geqslant 0$ or $\alpha_2 \leqslant 0$.
- There exists a sufficiently large constant M such that $0 \leqslant \frac{\alpha_{2j}}{\beta} \leqslant M$ for each j.

(4) Note that the last two conditions are similarly stated as in Case 3. In fact, we can reduce this case to Case 3.

Concluding remarks

Open problem 1

For more general S, is the S-CG closure still polyhedral?

We conjecture that our theorem can be extended to the case when S is the set of integer points contained in any rational polyhedron.

Open problem 2

We can define the S-split closure of a rational polyhedron as the set obtained after applying the intersection cuts from S-free splits.

Is the S-split closure of a rational polyhedron necessarily polyhedral as well? The first case to consider would be the case $S = \{0,1\}^n$.

References

[1] J. Dunkel and A. S. Schulz, A refined Gomory-Chvátal closure for polytopes in the unit cube, Technical report, March 2012, `http://www.optimization-online.org/DB_HTML/2012/03/3404.html`.

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dabeenl@andrew.cmu.edu