A Note on "A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs"

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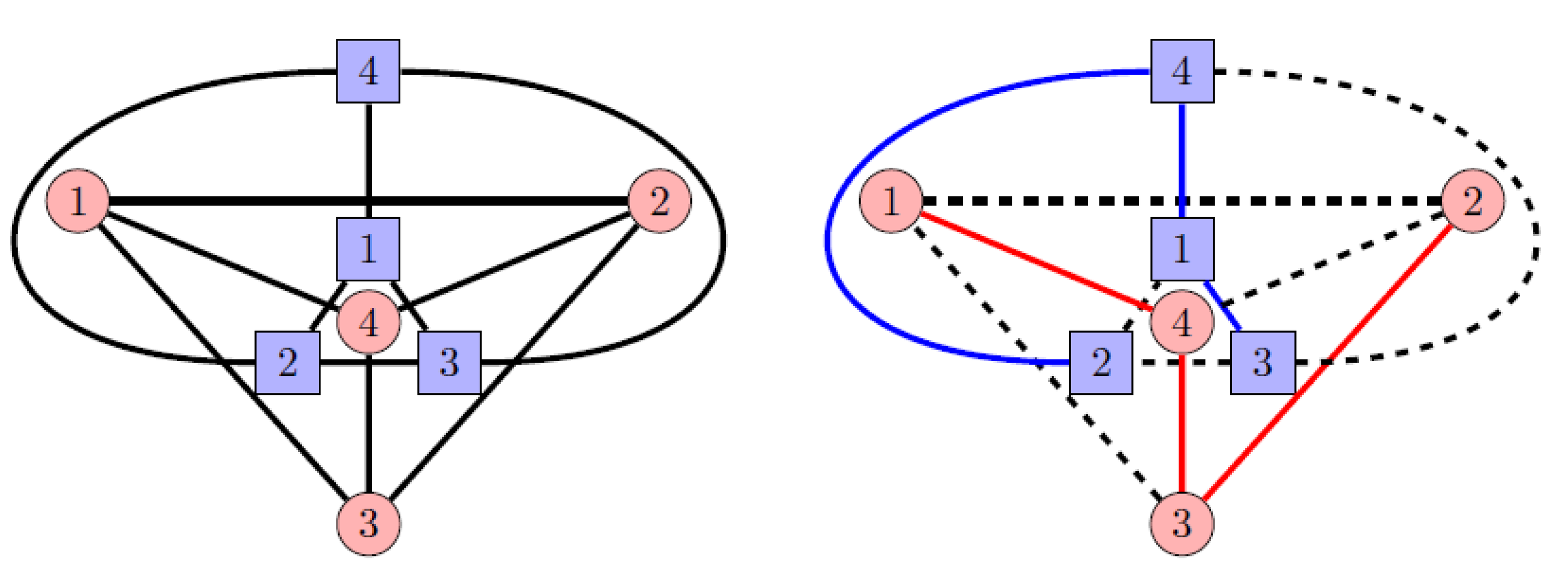
A histORy

- In 2002, Williams [1] proposed a linear-size and perfect extended formulation for spanning trees in planar graphs.
- The formulation exploits planarity to achieve size $O(n)$. Compare this to the best-known formulation for arbitrary graphs of size $O(mn)$.
- The formulation is employed in some other extended formulations (e.g. see Fiorini et al. [5]).
- In real-world applications, it is employed in vertex-induced connectivity [4], distribution system reconfiguration, political redistricting, and nature reserve networks.
- We show that Williams' spanning tree formulation is incorrect by constructing a binary feasible solution that does not represent a spanning tree. Fortunately, a small tweak corrects the formulation.

Complement Spanning Trees (See Tutte [3])

Suppose G is a connected planar graph. The edge subset $T \subseteq E$ is the edge set of a spanning tree of G if and only if $T^* := \{e^* \mid e \in E \setminus T\}$ is the edge set of a spanning tree of the dual graph G^*.

Spanning Trees in Planar Graphs



Notations

$G = (V, E)$	Primal multigraph
$G^* = (V^*, E^*)$	Dual multigraph
r	The primal root
r^*	The dual root
$\delta_H(i)$	Subset of edges incident to vertex i in graph H
$q_H(e)$	Set of the endpoints of edge e in graph H

Spanning Arborescences

A directed version of a spanning tree in which all edges are pointed away from a root vertex, in which case each vertex (besides the root) will have one incoming edge.

Variables

$$x_{e,i} = \begin{cases} 1, & \text{if primal edge } e \text{ is selected and oriented towards vertex } i \in V \\ 0, & \text{otherwise} \end{cases}$$

$$y_{e^*,u} = \begin{cases} 1, & \text{if dual edge } e^* \text{ is selected and oriented towards vertex } u \in V^* \\ 0, & \text{otherwise} \end{cases}$$

Williams' Spanning Tree Formulation

$$\sum_{e \in \delta_G(i)} x_{e,i} = 1 \qquad \forall i \in V \setminus \{r\} \qquad (1)$$

$$\sum_{e \in \delta_G(i)} x_{e,i} = 0 \qquad i = r \qquad (2)$$

$$\sum_{e^* \in \delta_{G^*}(u)} y_{e^*,u} = 1 \qquad \forall u \in V^* \setminus \{r^*\} \qquad (3)$$

$$\sum_{e^* \in \delta_{G^*}(u)} y_{e^*,u} = 0 \qquad u = r^* \qquad (4)$$

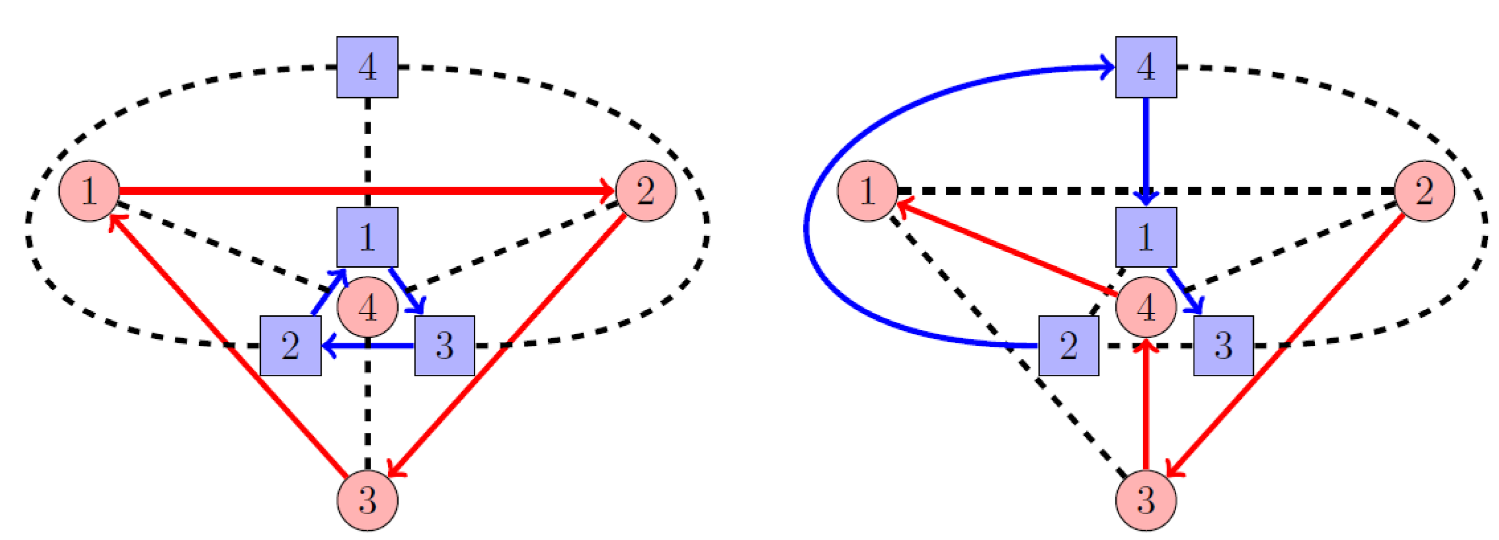
$$\sum_{i \in q_G(e)} x_{e,i} + \sum_{u \in q_{G^*}(e^*)} y_{e^*,u} = 1 \qquad \forall e \in E \qquad (5)$$

$$x_{e,i} \in \{0,1\} \qquad \forall e \in \delta_G(i),\ i \in V \qquad (6)$$

$$y_{e^*,u} \in \{0,1\} \qquad \forall e^* \in \delta_{G^*}(u),\ u \in V^*. \qquad (7)$$

Constraints (1) and (3) enforce that only one edge must enter each non-root vertex of primal and dual graphs, respectively. Constraints (2) and (4) enforce that no edge should enter the root vertex of primal and dual graphs, respectively. Constraints (5) imply that exactly one of the primal or dual crossing edges should be selected.

A Spanning Tree Counterexample



Root Rule (Due to Pashkovich)

Arbitrarily pick a face $r^* \in V^*$ as the dual root. Then, pick a vertex $r \in V$ from that same face to be the primal root.

Lemma (Due to Pashkovich)

If the Root Rule is followed, then Williams' spanning tree formulation is correct.

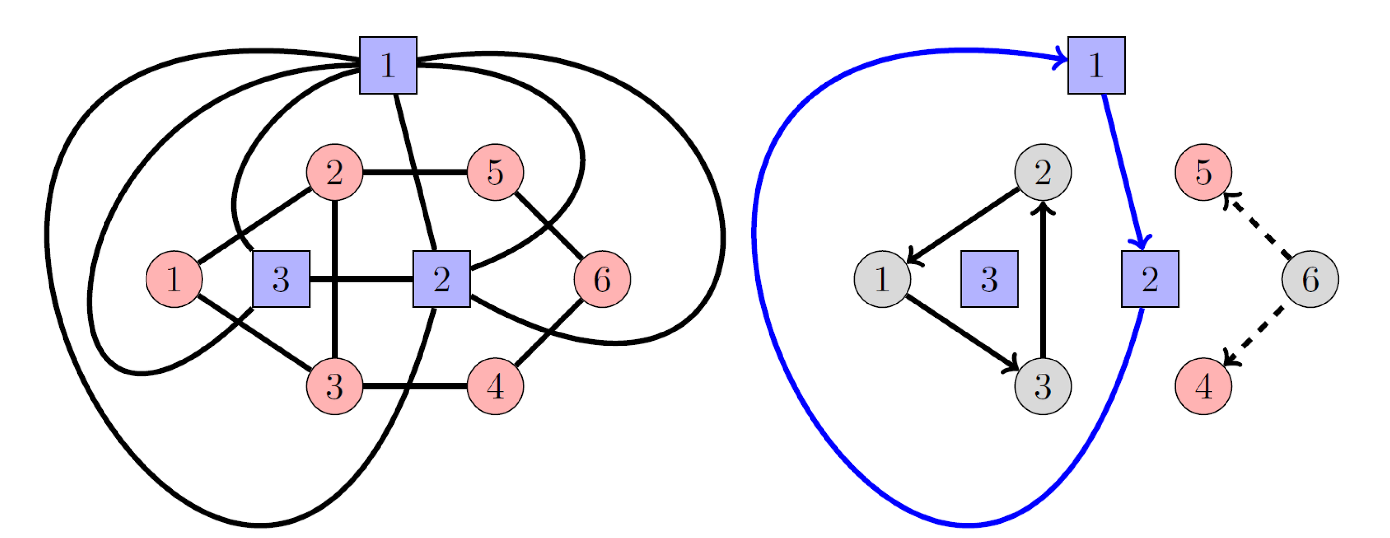
Theorem

Williams' spanning tree formulation is correct if and only if the Root Rule is followed.

Connected Subgraph Formulation

- In a second paper [4], Williams adapts his spanning tree formulation so that it selects contiguous parcels of land.
- We also provide a counterexample to disprove it. The same root rule fixes the flaw.

A Connected Subgraph Counterexample



References

[1] Justin C Williams. A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs. Networks, 39(1):53-60, 2002.
[2] Kanstantsin Pashkovich. Extended formulations for combinatorial polytopes. PhD thesis, Otto-von-Guericke-Universität Magdeburg, 2012.
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[4] Justin C Williams. A zero-one programming model for contiguous land acquisition. Geographical Analysis, 34(4):330-349, 2002.
[5] Samuel Fiorini, Tony Huynh, GwenaÂĺel Joret, and Kanstantsin Pashkovich. Smaller extended formulations for the spanning tree polytope of boundedgenus graphs. Discrete & Computational Geometry, 57(3):757-761, 2017.