Characterization and Approximation of Strong General Dual-Feasible Functions

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Introduction

Definitions

A function $\phi\colon D \to D$ is called a **(classical or general) Dual-Feasible Function** (cDFF or gDFF) if $\sum_{i\in I} x_i \le 1 \Rightarrow \sum_{i\in I} \phi(x_i) \le 1$ holds for any finite list of real numbers $x_i \in D$, where $D = [0,1]$ or $\mathbb{R}$.

DFFs have been used in several combinatorial optimization problems, including cutting stock problems. DFFs can derive feasible solutions to the dual problem of the LP relaxation efficiently, therefore providing fast dual bounds for the primal IP problem. DFFs can also generate valid inequalities for general IP problems.

The **maximal** (pointwise non-dominated) DFFs are of particular interest since they provide better bounds and stronger valid inequalities. A maximal DFF is said to be **extreme** if it can not be written as a convex combination of two other DFFs.

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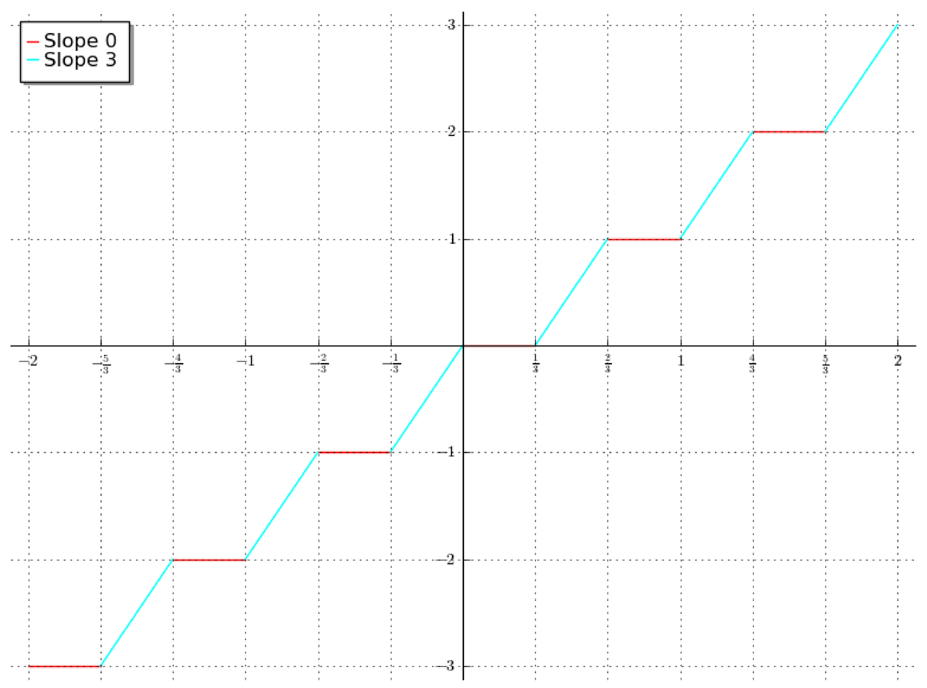

Figure: A continuous 2-slope extreme gDFF.

Characterization

Characterization of maximal gDFFs

A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is a maximal gDFF if and only if the following conditions hold:
(i) $\phi(0) = 0$. (ii) ϕ is superadditive. (iii) $\phi(x) \ge 0$ for all $x \in \mathbb{R}_+$. (iv) $\phi(x) = \alpha x$ for $0 \le \alpha < 1$ or $\phi(x) + \phi(1-x) = 1$.

The above theorem resolves the open question regarding the full characterization of maximal gDFFs proposed in the monograph [1]. Parallel to the restricted/strongly minimal functions in the Yıldız–Cornuéjols model [2], "restricted" and "strongly" maximal gDFFs can be defined by strengthening the notion of maximality.

We say that a gDFF ϕ is **implied via scaling** by a gDFF ϕ_1, if $\beta\phi_1 \ge \phi$ for some $0 \le \beta \le 1$. We call a gDFF ϕ **restricted maximal** if ϕ is not implied via scaling by a distinct gDFF ϕ_1.
Motivation: $\sum \phi(x) \le \sum \beta\phi_1(x) = \beta \sum \phi_1(x) \le \beta \le 1$.

We say that a gDFF ϕ is **implied** by a gDFF ϕ_1, if $\phi(x) \le \beta\phi_1(x) + \alpha x$ for some $0 \le \alpha, \beta \le 1$ and $\alpha + \beta \le 1$. We call a gDFF ϕ **strongly maximal** if ϕ is not implied by a distinct gDFF ϕ_1.
Motivation: $\sum \phi(x) \le \sum(\beta\phi_1(x) + \alpha x) = \beta \sum \phi_1(x) + \alpha \sum x \le \beta + \alpha \le 1$.

Characterization of restricted/strongly maximal gDFFs

A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is a restricted maximal gDFF if and only if the following conditions hold:
(i) $\phi(0) = 0$. (ii) ϕ is superadditive. (iii) $\phi(x) \ge 0$ for all $x \in \mathbb{R}_+$. (iv) $\phi(x) + \phi(1-x) = 1$.
A function $\phi\colon \mathbb{R} \to \mathbb{R}$ is a strongly maximal gDFF if and only if ϕ is a restricted maximal and $\lim_{\epsilon\to 0^+} \frac{\phi(\epsilon)}{\epsilon} = 0$.

Two Slope Theorem and Approximation Theorem

Theorem

1. Let ϕ be a continuous piecewise linear strongly maximal gDFF with only 2 slope values, then ϕ is extreme.
2. Let ϕ be a continuous restricted maximal gDFF, then for any $\epsilon > 0$, there exists an extreme gDFF ϕ_{ext} (= `two_slope_approximation_gdff_linear`(ϕ,ϵ)) such that $\|\phi - \phi_{\text{ext}}\|_\infty < \epsilon$.

A function name shown in typewriter font is the name of the function in our SageMath program [3]. The approximation is implemented for piecewise linear functions with finitely many pieces.

Example

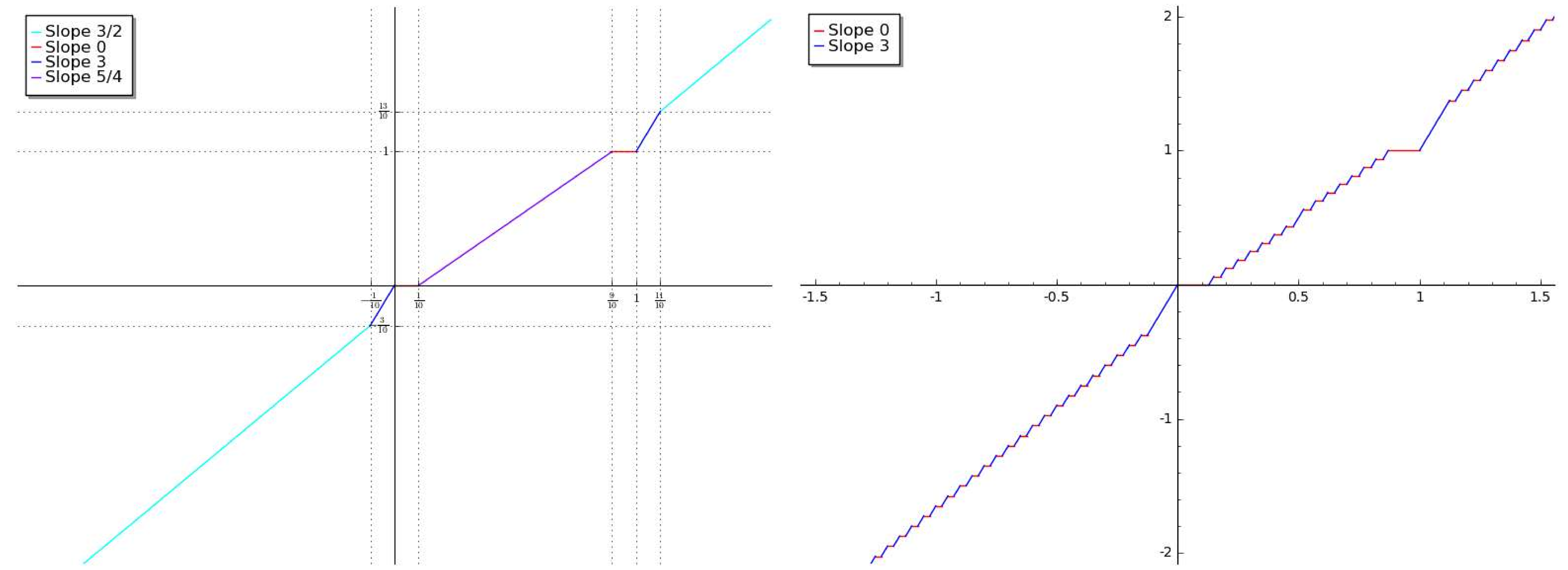

Figure: The function $\phi_{s,\delta}$ on the left is generated by the Python function `phi=phi_s_delta(delta=1/10,s=3/2)`. The function on the right generated by `phi_ext=two_slope_approximation_gdff_linear(phi,epsilon=10)` is a 2-slope extreme gDFF, and `phi_ext` approximates the function `phi` in $\|\cdot\|$-norm to error $< \frac{1}{20}$.

Relation to Cut-Generating Functions

Conversion between gDFFs and cut-generating functions

Given a valid/maximal/restricted maximal gDFF ϕ, then for every $0 < \lambda < 1$, the following function is a valid/minimal/restricted minimal cut-generating function for $Y_{=1} = \{y : \sum_{r\in\mathbb{R}} r\, y(r) = 1,\ y\colon \mathbb{R} \to \mathbb{Z}_+ \text{ and } y \text{ has finite support}\}$:

$$\pi_\lambda(x) = \frac{x - (1-\lambda)\,\phi(x)}{\lambda}.$$

Given a valid/minimal/restricted minimal cut-generating function π for $Y_{=1}$, which is Lipschitz continuous at $x = 0$, then there exists $\delta > 0$ such that for all $0 < \lambda < \delta$ the following function is a valid/maximal/restricted maximal gDFF:

$$\phi_\lambda(x) = \frac{x - \lambda\,\pi(x)}{1-\lambda}, \quad 0 < \lambda < 1.$$

Example

Consider the lattice points in the first quadrant. Let π' be the Gomory Mixed Integer Cut with $f = \frac{1}{2}$, and it is not hard to show that $\pi(x) = \pi'(\frac{11}{2}x)$ is a valid cut-generating function for $Y_{=1}$. In particular, π generates the valid inequality $y \ge 1$ (green), which separates the origin, for lattice points on the red line $\{(x,y) \in \mathbb{R}^2_+ : \frac{2}{11}x + \frac{3}{11}y = 1\}$.
Choose $\lambda = \frac{1}{11}$, then the gDFF $\phi(x) = \frac{x - \lambda\,\pi(x)}{1-\lambda}$ generates the valid inequality $\frac{1}{5}x + \frac{1}{5}y \le 1$ (blue), which does not separate the origin, for lattice points in the shaded region $\{(x,y) \in \mathbb{R}^2_+ : \frac{2}{11}x + \frac{3}{11}y \le 1\}$.
Therefore we say $y \ge 1$ can be lifted to $\frac{1}{5}x + \frac{1}{5}y \le 1$.
The conversion involves adding a multiple of the defining equality $\sum_{r\in\mathbb{R}} r\, y(r) = 1$ to a valid inequality, which is called "tilting" by Aráoz et al.

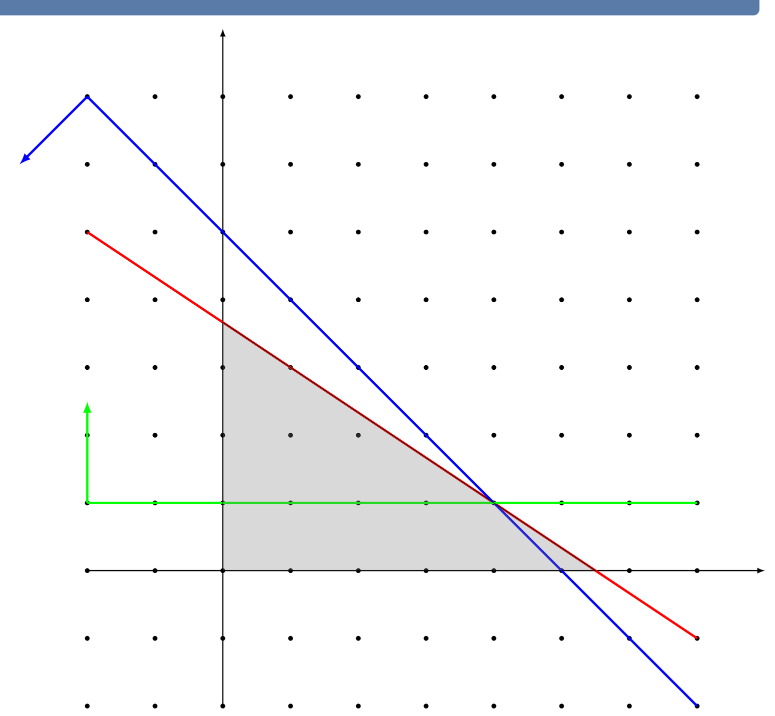

Figure: Lifting of inequalities.

References

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(Version 1.0).