Legal Assignments, the EADAM algorithm

Yuri Faenza, Xuan Zhang
Columbia University

COLUMBIA UNIVERSITY

Background

- In New York City, about 100,000 students enter public high schools each year.
- How to assign students to schools?

One-to-Many Matching Model

- Every student $a \in A$ has a strict **preference** ordering, $>_a$, of the schools (possibly incomplete).
- Every school $b \in B$ has a strict **preference** ordering, $>_b$, of the students (possibly incomplete) and a **quota** q_b.
- Represent an instance as $(G(A \dot\cup B, E), <, \mathbf{q})$.

Objectives

- **Stability**
  - no *blocking pairs*, i.e. no student and school that are not assigned to each other would both prefer to be.
- **Pareto efficiency (for students)**
  - no assignment where every student is at least as good, and some student is strictly better off.
- **Legality** [5]
  - no blocking pair that is *redressable*, i.e. the student and school forming the blocking pair are not matched in any legal assignments.

Trade-off and Why EADAM

- There is a significant trade-off between stability and efficiency.
- Gale-Shapley [2] outputs a stable assignment that is optimal for the students, but may not be Pareto efficient.
- Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) asks for students' *consent*, as to waive his priority to a certain school if applying only interrupts other students' chance of being admitted, but at no gain to himself.
- If all students consent, output of EADAM is Pareto efficient; otherwise, the output is *constraint efficient* [6]:
  - **Pseudo stable**: with consent, it respects all students' priorities.
  - **Pseudo efficient**: among all assignments that respect students' priorities, it is optimal for the students.

What is Known

- The set of legal assignments exists and is unique.
- The set of legal assignments forms a lattice, which has the set of stable assignments as a sublattice.
- The student-optimal legal assignment coincides with the output of EADAM when all students consent, thus is Pareto efficient.

What is New

- **Structural:**
  - The set of legal assignments coincides with the set of stable assignments in a (sub)instance, which we call the *legalized* instance.
- **Algorithmic #1:**
  - The legalized instance, the student-optimal legal assignment, and the school-optimal legal assignment can be found in time $O(|E|)$.
  - Legal assignment with maximum weight can be found in polynomial time.
- **Algorithmic #2:**
  - Output of EADAM with consent, with *any* set of students consenting, can be found in time $O(|E|)$.

Techniques

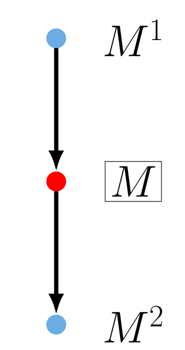
- Generalize *rotations* (trading cycles that preserve stability) to one-to-many settings.
- Jump out of the stable lattice by identifying and removing edges that are not in the legalized instance.
- Construct *rotation digraphs* locally and partially for fast implementation.

Example

Consider the following instance with 6 students and 3 schools. Each school has a quota of 2.

$a_1 : \boxed{b_2} > b_3 > b_1$
$a_2 : b_1 > \boxed{b_2} > b_3$
$a_3 : \mathbf{b_3} > \boxed{b_1} > b_2$
$a_4 : \boxed{b_1} > b_2 > b_3$
$a_5 : \boxed{b_3} > b_2 > b_1$
$a_6 : \mathbf{b_1} > \boxed{b_3} > b_2$

$b_1 : \underline{a_1} > \boxed{a_4} > \boxed{a_3} > a_5 > a_2 > \mathbf{a_6}$
$b_2 : \underline{a_3} > \boxed{a_2} > a_6 > \boxed{a_1} > a_5 > a_4$
$b_3 : \boxed{a_6} > a_1 > \boxed{a_5} > a_2 > a_4 > \mathbf{a_3}$

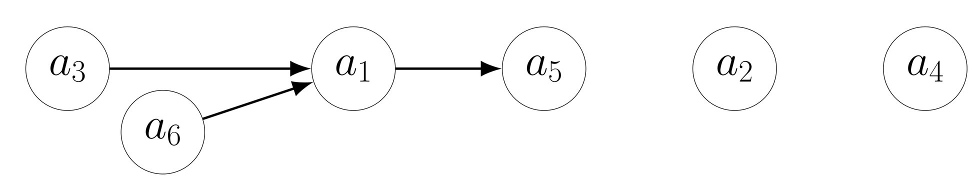


This instance has one stable assignment $\boxed{M} = \{a_1b_2, a_2b_2, a_3b_1, a_4b_1, a_5b_3, a_6b_3\}$, and two additional legal assignments: $M^1 = \{a_1b_2, a_2b_2, \mathbf{a_3b_3}, a_4b_1, a_5b_3, \mathbf{a_6b_1}\}$ and $M^2 = \{\underline{a_1b_1}, a_2b_2, \underline{a_3b_2}, a_4b_1, a_5b_3, a_6b_3\}$, which can be obtained via ...

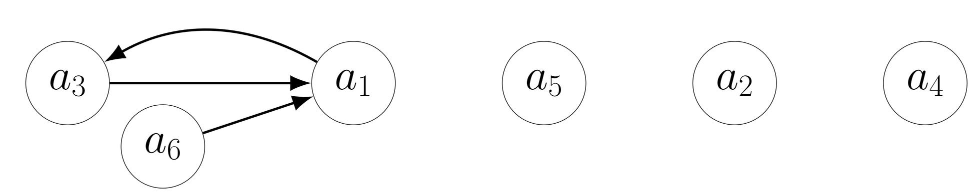
`Rotate-and-Remove`

goal: school-optimal legal assignment M^2

1. let $s^*_M(a)$ be the first school $b \neq M(a)$ on a's preference list that prefers a to some of her assigned students
2. point a to $s^*_M(a)$'s least preferred student in M, as to construct the rotation digraph D_A



3-1. if $(a', a) \in A(D_A)$ and a is a sink, remove $a'M(a)$ from the instance and repeat



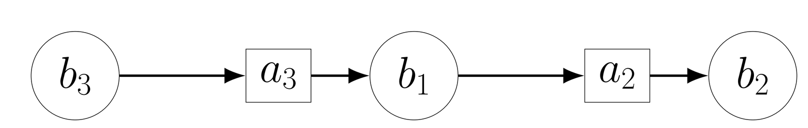
3-2. if D_A has a cycle C, for every $(a', a) \in A(C)$, reassign a' to $M(a)$ to obtain a new assignment, M^2 in this case; repeat

4. execute until D_A only has isolated nodes

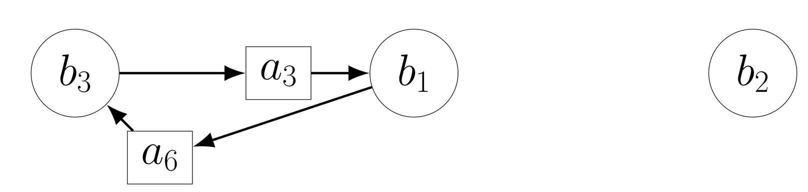
`Reverse Rotate-and-Remove`

goal: student-optimal legal assignment M^1

1. let $s^*_M(b)$ be the first student $a \notin M(b)$ on b's preference list that prefers b to his assigned school
2. point b to $s^*_M(b)$ and point $s^*_M(b)$ to $M(s^*_M(b))$, as to construct the rotation digraph D_B



3-1. if (b', a) and $(a, b) \in A(D_B)$ and b is a sink, remove ab' from the instance and repeat



3-2. if D_B has a cycle C, for every $(b', a) \in A(C)$, reassign a to b' to obtain a new assignment, M^1 in this case; repeat

4. execute until D_B only has isolated nodes

Reverse Rotate-and-Remove with Consent

Fast Implementation of EADAM

- In step 3-1. of `reverse rotate-and-remove`, if a is nonconsenting, we will additionally remove all edges $a'b'$ such that $a >_{b'} a'$.

Figure 1: All students consent

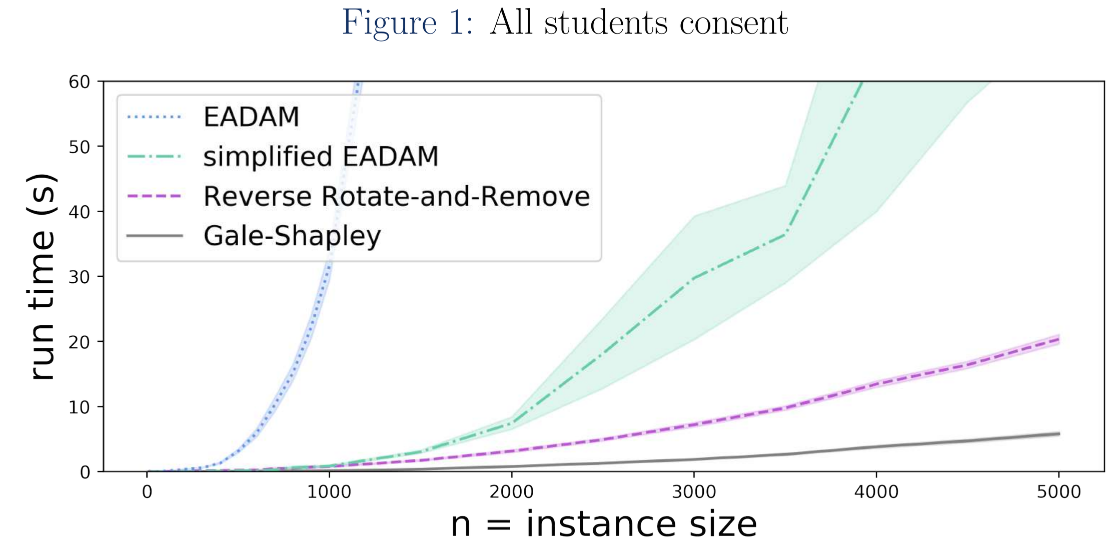
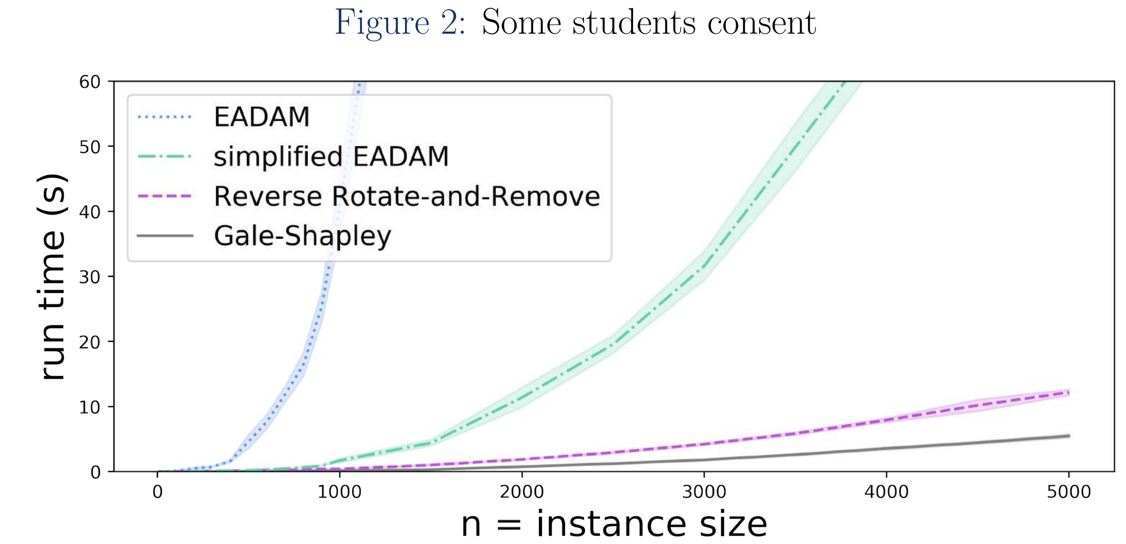


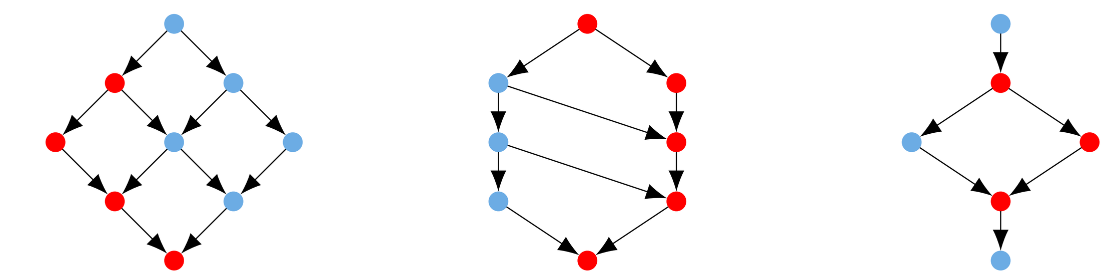
Figure 2: Some students consent



Lattice

- Stable assignment M *dominates* stable assignment M', denoted by $M \succeq M'$, if $M(a) \geq_a M'(a)$ for every student a.
- The set of stable assignments $\mathcal{S}$, with dominance relation $\succeq$, forms a *distributive lattice*, with *join* ($\vee$) and *meet* ($\wedge$):
  - $M_1 \vee M_2 := \{ab : a \in A, b = M_1(a) \vee_{>_a} M_2(a)\}$
  - $M_1 \wedge M_2 := \{ab : a \in A, b = M_1(a) \wedge_{>_a} M_2(a)\}$
- The set of legal assignments $\mathcal{L}$, with dominance relation $\succeq$, forms a distributive lattice with the same join and meet.

Examples of Lattices



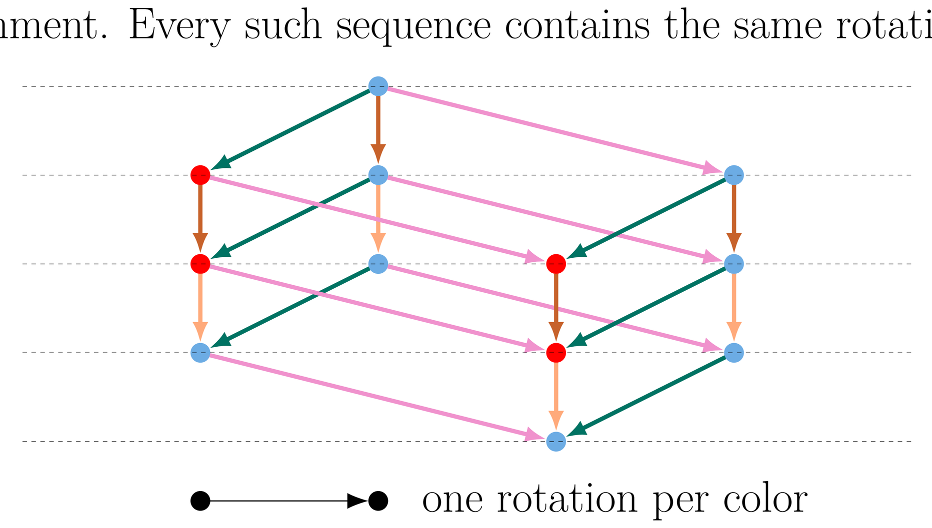
● stable assignments ● unstable legal assignments

Rotations

- A cycle $\rho = b_0, a_0, b_1, a_1, \cdots, b_{r-1}, a_{r-1}$ is a *student-rotation exposed* at stable assignment M if $a_ib_i \in M$ and $b_{i+1} = s^*_M(a_i)$ for all i, with indices taken modulo r.
- We can move down the lattice via *rotation eliminations*. Obtain an assignment M' that is immediately below M in the lattice, by assigning

$$M'(a) = \begin{cases} M(a) & \text{if } a \notin \rho \\ b_{i+1} & \text{if } a = a_i \end{cases}$$

- Every stable assignment can be generated by a sequence of rotation eliminations, starting from the student-optimal stable assignment. Every such sequence contains the same rotations.



- Concepts and results can be extended to school-rotations.

References

[1] Faenza, Y., & Zhang, X. (2018). Legal assignments and fast EADAM with consent via classical theory of stable marriages. Working paper.

[2] Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly, 69*(1), 9-15.

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[4] Kesten, O. (2010). School choice with consent. *The Quarterly Journal of Economics, 125*(3), 1297-1348.

[5] Morrill, T. (2016). Which School Assignments Are Legal?. Working paper, North Carolina State University, 2016.

[6] Tang, Q., & Yu, J. (2014). A new perspective on Kesten's school choice with consent idea. *Journal of Economic Theory, 154*, 543-561.