Valid Inequalities for Mixed Integer Bilevel Linear Optimization Problems

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Contributions

- Comparing the performance of the known classes of valid inequalities for mixed integer bilevel linear optimization problems (MIBLPs) within the MiBS open source solver [6], [3]
- Suggestion of a new method that can be used to
  - strengthen the known cuts by generating supporting valid inequalities
  - compare the strength of various valid inequalities
  - develop a new family of cuts for MIBLPs

Problem Definition

The general form of an MIBLP is

\[
\min \{ cx + d^Ty \mid x \in X, y \in P_1(x) \cap P_2(x) \cap Y, d^Ty \leq \phi(b^2 - A^2x) \},
\]

where

- First-level feasible region
  \( P_1(x) = \{ y \in \mathbb{R}^n_+ \mid A^1 x + G^1 y \geq b^1 \} \)
- Second-level feasible region
  \( P_2(x) = \{ y \in \mathbb{R}^m_+ \mid G^2 y \geq b^2 - A^2 x \} \)
- Second-level value function
  \( \phi(\beta) = \min\{d^Ty \mid G^2 y \geq \beta, y \in Y\} \quad \forall \beta \in \mathbb{R}^m_+ \) (VF)

Integrality Constraints: Sets \( X = \mathbb{Z}^n_+ \times \mathbb{R}^{n_1-1}_+ \) and \( Y = \mathbb{Z}^m_+ \times \mathbb{R}^{m_1-1}_+ \) represent integrality constraints.

Further related definitions are as below.

- Linking Variables: The set of indices of first-level variables with non-zero coefficients in the second-level problem (\( x_L \)).
- Bilevel feasible region
  \( F = \{ (x, y) \in X \times Y \mid y \in P_1(x) \cap P_2(x), d^Ty \leq \phi(b^2 - A^2x) \} \)

Relaxation Problem

- Removing the optimality constraint of the second-level problem
  \( S = \{ (x, y) \in \mathbb{R}^{n+m}_+ \mid x \in X, y \in P_1(x) \cap P_2(x) \cap Y \} \)

- Removing the optimality constraint of the second-level problem and the integrality constraints
  \( P = \{ (x, y) \in \mathbb{R}^{n+m}_+ \mid y \in P_1(x) \cap P_2(x) \} \)

MiBS uses \( P \) as the relaxation problem.

Valid Inequalities

The set of valid inequalities for MIBLPs can be classified based on the goals of generating them.

- Feasibility cuts
  - Goal: Removing \((x, y) \notin X \times Y\)
  - These valid inequalities are violated by \((x, y)\), but they are valid for \( S \).
  - This set includes all valid inequalities work for the mixed integer optimization problems (MIBLPs).

- Optimality cuts
  - Goal: Removing \( (x, y) \notin X \times Y \), but \( d^Ty > \phi(b^2 - A^2x) \)
  - These valid inequalities are violated by \((x, y)\), but they are valid for \( F \).
  - This set includes
    - Intenders cut
      - \( -x_L \leq b^1 \)
      - \( A^1 x + G^1 y \geq b^1 \) corresponds to each linking variable \( x_L \), there exists \( y \), so that \( x_L = 1 \) results \( y = 0 \).
      - \( G^2 y \leq 0 \).
    - Intersection cut (types I and II) [4]
      - \( A^1 x + G^1 y \geq \beta \) in all \((x, y)\) \( y \) \( S \).
      - \( -Y \) \( \geq 2^n \) and \( \phi \) \( 2^n \).
    - Integer no-good cut [6]
      - \( -x_L \leq b^1 \)
    - Watermelon intersection cut [4]
      - \( A^1 x + G^1 y \geq \beta \) in all \((x, y)\) \( y \) \( S \).
      - \( -A^1 x - G^1 y \geq \beta \) in all \((x, y)\) \( y \) \( S \).
    - Hypercube intersection cut [4]
      - \( -x_L \leq b^1 \)

- Projected optimality cuts
  - Goal: Removing all \((x, y) \in P \) with \( x_L = 0 \)
  - These valid inequalities are violated by \((x, y) \in P \) with \( x_L = 0 \), but are valid for \( \alpha = \min \{ d^Ty \mid (x, y) \in F \} \)
  - This set includes the generalized no-good cut which works for the problems with \( x_L \leq b^1 \).