Valid Inequalities for Mixed Integer Bilevel Linear Optimization Problems

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Contributions

- Comparing the performance of the known classes of valid inequalities for mixed integer bilevel linear optimization problems (MIBLPs) within the `MibS` open source solver [6],[3]
- Suggestion of a new method that can be used to
  - strengthen the known cuts by generating supporting valid inequalities
  - compare the strength of various valid inequalities
  - develop a new family of cuts for MIBLPs

Problem Definition

The general form of an MIBLP is

$$\min\left\{cx + d^1y \,\middle|\, x \in X, y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \cap Y, d^2y \le \phi(b^2 - A^2x)\right\}, \quad \text{(MIBLP)}$$

where

- **First-level feasible region**

$$\mathcal{P}_1(x) = \left\{y \in \mathbb{R}^{n_2}_+ \,\middle|\, A^1x + G^1y \ge b^1\right\}$$

- **Second-level feasible region**

$$\mathcal{P}_2(x) = \left\{y \in \mathbb{R}^{n_2}_+ \,\middle|\, G^2y \ge b^2 - A^2x\right\}$$

- **Second-level value function**

$$\phi(\beta) = \min\{d^2y \mid G^2y \ge \beta, y \in Y\} \quad \forall \beta \in \mathbb{R}^{m_2}. \quad \text{(VF)}$$

- **Integrality Constraints**: Sets $X = \mathbb{Z}^{r_1}_+ \times \mathbb{R}^{n_1-r_1}_+$ and $Y = \mathbb{Z}^{r_2}_+ \times \mathbb{R}^{n_2-r_2}_+$ represent integrality constraints.

Further related definitions are as below.

- **Linking Variables:** The set of indices of first-level variables with non-zero coefficients in the second-level problem (x_L).
- **Bilevel feasible region**

$$\mathcal{F} = \left\{(x,y) \in X \times Y \,\middle|\, y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x), d^2y \le \phi(b^2 - A^2x)\right\}$$

- **Relaxation Problem**

❶ Removing the optimality constraint of the second-level problem

$$\mathcal{S} = \left\{(x,y) \in \mathbb{R}^{n_1 \times n_2}_+ \,\middle|\, x \in X, y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \cap Y\right\}$$

❷ Removing the optimality constraint of the second-level problem and the integrality constraints

$$\mathcal{P} = \left\{(x,y) \in \mathbb{R}^{n_1 \times n_2}_+ \,\middle|\, y \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x)\right\}$$

`MibS` uses $\mathcal{P}$ as the relaxation problem.

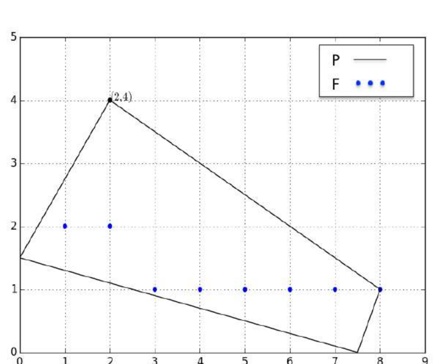

Figure: The feasible region of IBLP[5]

Valid Inequalities

The set of valid inequalities for MIBLPs can be classified based on the goals of generating them.

- **Feasibility cuts**
  - **Goal:** Removing $(\bar{x}, \bar{y}) \notin X \times Y$
  - These valid inequalities are violated by $(\bar{x}, \bar{y})$, but they are valid for $\mathcal{S}$.
  - This set includes all valid inequalities work for the mixed integer optimization problems(MIPs).
- **Optimality cuts**
  - **Goal:** Removing $(\bar{x}, \bar{y}) \in X \times Y$, but $d^2\bar{y} > \phi(b^2 - A^2\bar{x})$
  - These valid inequalities are violated by $(\bar{x}, \bar{y})$, but they are valid for $\mathcal{F}$.
  - This set includes

❶ **Benders cut**
Assumptions
- $x_L \subseteq \mathbb{B}^L$.
- Corresponding to each linking variable x_i, there exists y_i so that $x_i = 1$ results $y_i = 0$.
- $G^2 \le 0$.

❷ **Increasing objective cut**[1]
Assumptions
- $x_L \subseteq \mathbb{B}^L$.
- $A^2 \le 0$.

❸ **Intersection cut(types I and II)**[4]
Assumptions
- $A^2x + G^2y - b^2 \in \mathbb{Z}$ for all $(x,y) \in \mathcal{S}$.
- $Y = \mathbb{Z}^{n_2}$ and $d^2 \in \mathbb{Z}^{m_2}$.

❹ **Integer no-good cut**[2]
Assumptions
- $X = \mathbb{Z}^{n_1}$ and $Y = \mathbb{Z}^{n_2}$.
- b^1, b^2, A^1, A^2, G^1 and G^2 are discrete.

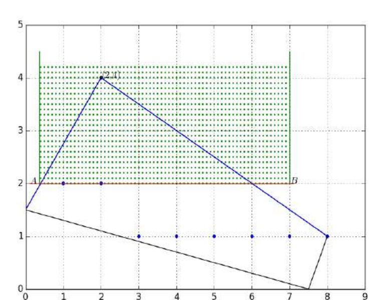

Figure: Intersection cut

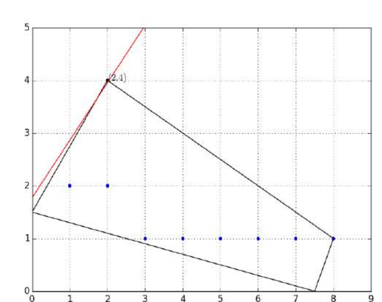

Figure: Integer no-good cut

❺ **Watermelon intersection cut**[4]
Assumptions
- $A^2x + G^2y - b^2 \in \mathbb{Z}$ for all $(x,y) \in \mathcal{S}$.
- $d^2 \in \mathbb{Z}^{m_2}$.

❻ **Hypercube intersection cut**[4]
Assumption
- $x_L \subseteq \mathbb{Z}^L$.

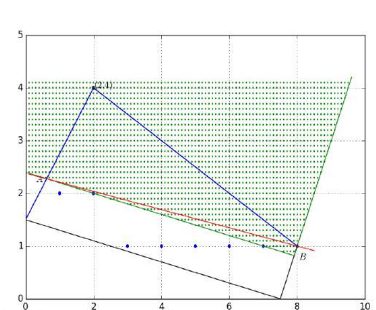

Figure: Watermelon intersection cut

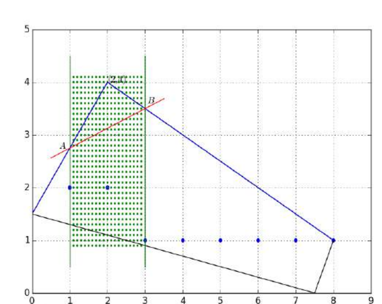

Figure: Hypercube intersection cut

- **Projected optimality cuts**
  - **Goal:** Removing all $(x,y) \in \mathcal{P}$ with $x_L = \lambda \in \mathbb{Z}^L$
  - These valid inequalities are violated by $(x,y) \in \mathcal{P}$ with $x_L = \gamma \in \mathbb{Z}^L$, but are valid for $\text{conv}\left(\{(x,y) \in \mathcal{F} \mid cx + d^1y < U\}\right)$, where U represents the current incumbent.
  - This set includes the **generalized no-good cut** which works for the problems with $x_L \subseteq \mathbb{B}^L$.

References

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6.Tahernejad, S., Ralphs, T. K., and DeNegre, S. T. (2016). A branch-and-cut algorithm for mixed integer bilevel linear optimization problems and its implementation. Tech. rep., COR@ L Laboratory Technical Report 16T-015-R3, Lehigh University.

Strengthening the Valid Inequalities

- **Idea:** The valid inequality $\alpha x \ge \beta$ can be strengthened by increasing β.
- The best value of rhs can be obtained by solving

$$\min_{(x,y) \in \mathcal{F}^t} \alpha x \quad (1)$$

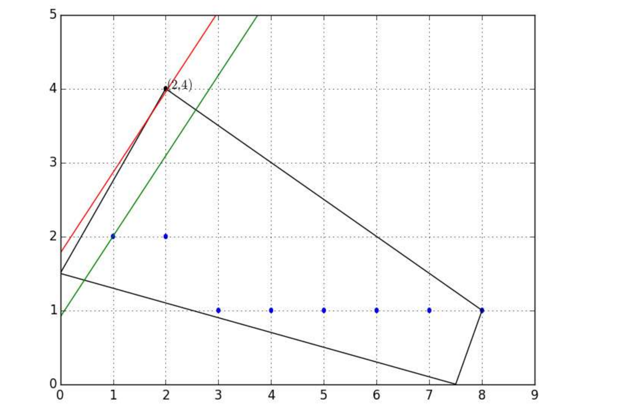

Figure: Improving integer no-good cut

- The strength of different cuts can be compared by measuring the difference between the rhs of each cut with its best value.
- **Mixed integer no-good cut**
  - The left-hand-side employs the idea of int no-good cut.
  - The rhs is obtained by solving (1).
  - The only assumption is $x_L \subseteq \mathbb{Z}^L$.

Computational Results

Two data sets were employed in experiments: `INTERD-DEN`, `IBLP-DEN`

Performance of different cuts

- Data set: `INTERD-DEN`

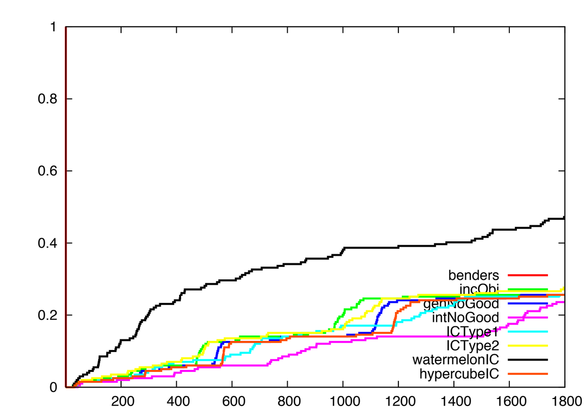

Figure: Perf profile for solution time

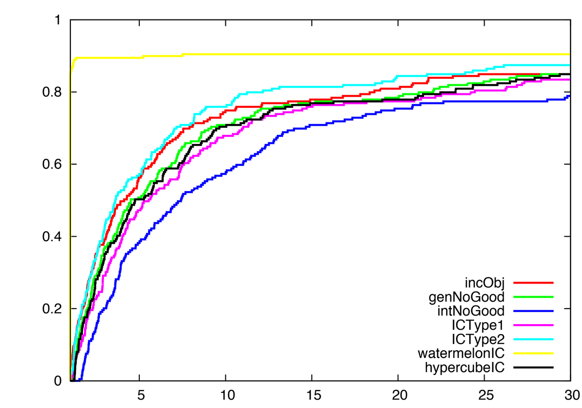

Figure: Perf profile for solution time (without benders cut)

Data set: `IBLP-DEN`

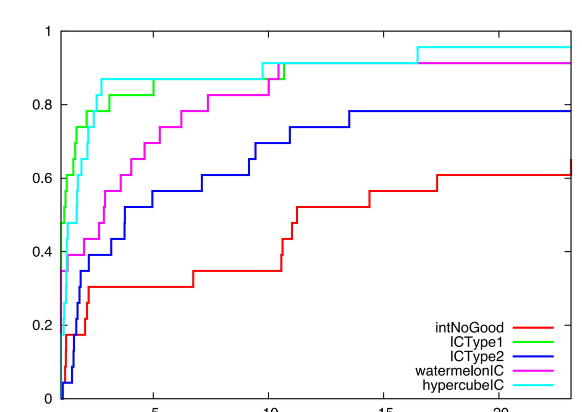

Figure: Perf profile for solution time

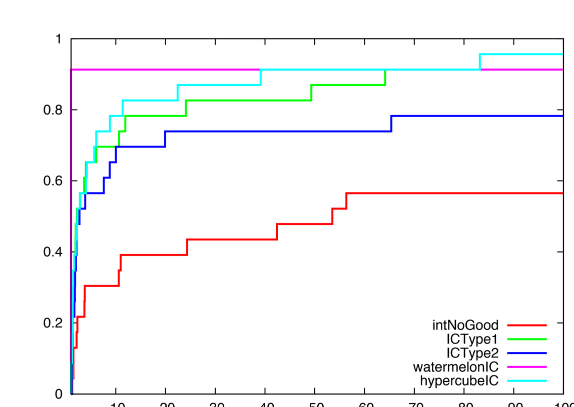

Figure: Perf profile for number of nodes

Impact of improving the rhs of integer no-good cut

- Data set: `INTERD-DEN`

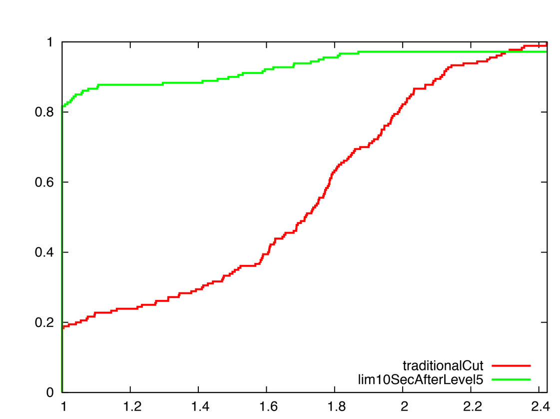

Figure: Perf profile for solution time

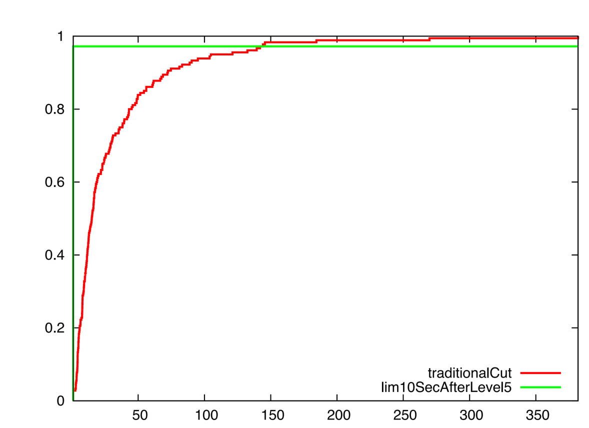

Figure: Perf profile for number of nodes